Optimal Control book

chapter 10.

ELEC 5530/6530
Robot Control
Lecture 9: Optimal Control

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Introduction to Optimal Control

Choose a control input over time so a dynamic system behaves in the best possible way.

Core idea

Given system dynamics **$\dot{x} = f(x,u,t)$**, find the control **u(t)** that minimizes cost (or maximizes reward) while satisfying constraints.

$$\min_{u(t)} \; J = \phi(x(T)) + \int_0^T L(x(t), u(t), t)\, dt$$

Typical ingredients

- State x(t): variables describing the system
- Control u(t): decision you can manipulate
- Dynamics: how the state evolves over time
- Cost function: what "best" means
- Constraints: limits on states or controls

$|x| < \bar{x}$

$|u| < \bar{u}$

How to think about it

Model
Dynamics
$\dot{x}=f(x,u,t)$

→ **Controller**
Choose
u(t)

→ **Goal**
Minimize
cost

Constraints
limits on states,
controls, time

Applications

Robotics • aerospace • energy • finance • process control

Common solution tools: Pontryagin's Maximum Principle, dynamic programming, and MPC.

Why do we care about Optimal Control

It connects powerful tools

- Adaptive Dynamic Programming
- Reinforcement Learning
- Model Predictive Control

It applies for broad areas

- Robot coordination control
- Energy regulation
- Industry process control

Optimal Control of robot motion

$$M\ddot{q} + V + G = \tau$$

Tracking error

$$e = q_d - q$$

$$\dot{e} = \dot{q}_d - \dot{q}$$

$$\ddot{e} = \ddot{q}_d - \ddot{q}$$

$$\ddot{e} = \ddot{q}_d + M^{-1}\underline{(V + G - \tau)}$$

Define control input

$$u \triangleq \ddot{q}_d + M^{-1}(V + G - \tau)$$

$$\ddot{e} = u$$

State-space representation

Define $\underline{\underline{x}} = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$

$$\dot{\underline{\underline{x}}} = \begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$\dot{x} = Ax + Bu$$

Problem: Find u to minimize a cost J in terms of $u = -Kx$

Define cost function

$$V(x(t)) = \int_t^{\infty} (x^T Q x + u^T R u)\, d\tau$$

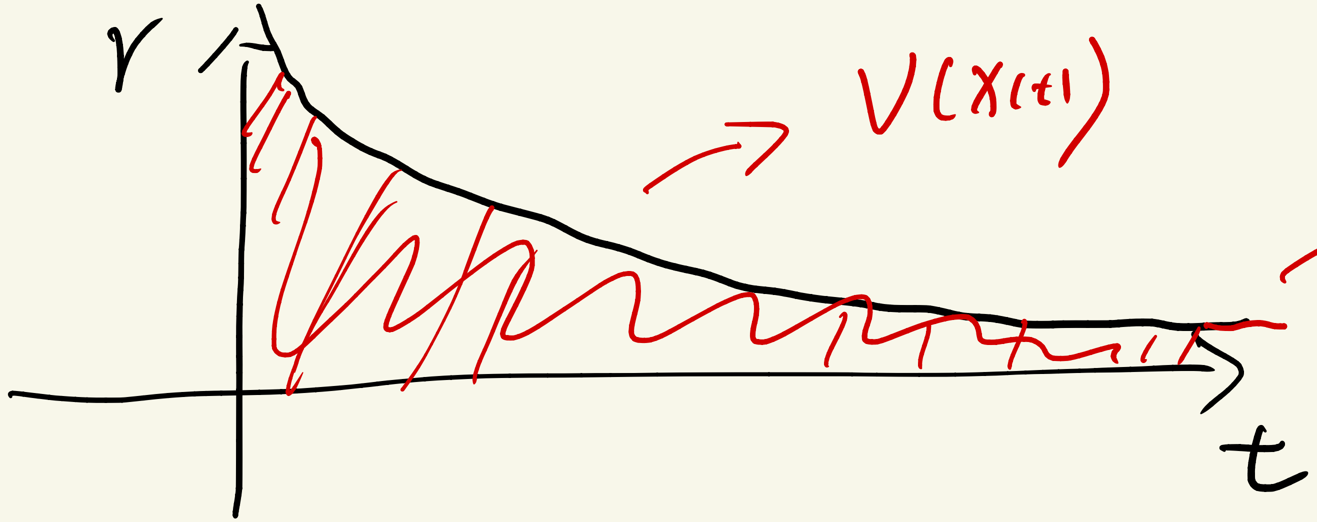
$$\int_t^{\infty} (x^T Q x + x^T K^T R K x)\, dt$$

$Q \geq 0 \quad , \quad R > 0$

$= x^T(\ ?\ )x$

> 0

$r = x^T Q x + u^T R u \geq 0$



go to zero
because u
stablizes x

How to find optimal u^* (K^*)

1. Define Hamilton function H

$$H(x, \nabla V, u) = \underline{x^TQx + u^TRu} + \dot{V} \rightarrow \frac{dV}{dt}$$

$$= x^TQx + u^TRu + \frac{\partial V}{\partial x}\dot{x}$$

$$= x^TQx + u^TRu + \left(\frac{\partial V}{\partial x}\right)^T (Ax+Bu)$$

$\uparrow u^* \uparrow$ $\uparrow$

2. Pontryagin's minimal principle

$$u^* \triangleq \boxed{\frac{\partial H}{\partial u} = 0}$$

$$\frac{\partial H}{\partial u} = \underline{2u^T R} + \underline{\left(\frac{\partial V}{\partial x}\right)^T B} = 0$$

$$u^* = -\frac{1}{2} R^{-1} B^T \frac{\partial V}{\partial x}$$ → problem

3. $V(x)$ can be represeneed by

$$V(x) = x^T P x, \quad P > 0$$

$\rightarrow P = P^T$

4. $$u^* = -\frac{1}{2} R^{-1} B^T \, 2 P x$$

$$= -\underbrace{R^{-1}}_{m\times m} \underbrace{B^T}_{m\times n} \underline{\underline{P}} \underbrace{x}_{n\times 1}$$

$\rightarrow$ unknown, to be solved

$K^* = R^{-1} B^T P$

$$\frac{\partial V}{\partial x} = \frac{\partial (x^T P x)}{\partial x}$$

$$= \underbrace{\underbrace{2x^T P}_{1 \times n}}_{} \quad ? \quad \underbrace{\underbrace{2 \underset{n\times n}{P} \underset{n\times 1}{x}}_{n\times 1}}_{} \quad \checkmark$$

we pick this because it should match dimension in

$$-R^{-1}B^T(Px)$$

5. Replace u with $u^* = -\underline{R^{-1}B^T}\overset{K^*}{P}x$ in Hamilton function $H(x, u, \nabla V) = 0$ to yield

$$x^TQx + (-R^{-1}B^TPx)^T R(-R^{-1}B^TPx)$$

$$+ (2Px)^T(Ax - BR^{-1}B^TPx) = 0$$

$$\underline{x^TQx} + \underline{}\, x^TPBR^{-1}B^TPx + \underline{2x^TPAx}$$

$$- \underline{2}\, x^TPBR^{-1}B^TPx = 0$$

$$x^TPAx$$

$$= x^TA^TPx$$

$$\boxed{x^TQx - x^TPBR^{-1}B^TPx + x^TPAx + x^TA^TPx = 0}$$

Remove x in ~~both~~ side ~~to~~ yield Algebraic Riccati equation (ARE)

$$A^TP + PA - \underline{PBR^{-1}B^TP} + Q = 0$$

$$\mathrm{care}(A, B, Q, R)$$

$$\to P$$

$$\to K^* = R^{-1}B^T P$$

Remark: Replace u with $u^* = -K^* x$ in $H = 0$

$$x^T Q x + (-K^* x)^T R (-K^* x) + \underline{(2Px)^T (Ax - BK^* x)} = 0$$

$$x^T Q x + x^T (K^*)^T R K^* x$$

$$+ \underline{(Px)^T (Ax - BK^* x)} + \underline{[(Px)^T (Ax - BK^* x)]^T} = 0$$

$$x^T Q x + x^T (K^*)^T R K^* x$$

$$\underline{+ x^T P A x - x^T \underline{PB} K^* x + x^T A P x} - x^T (K^*)^T \underline{B^T P} x = 0$$

$$K^* = R^{-1} B^T P \Rightarrow R K^* = B^T P \quad \left(PB = (K^*)^T R \right)$$

$$x^T Q x \underset{=}{+} x^T (K^*)^T R K^* x$$

$$+\ x^T P A x + x^T A^T P x - 2x^T (K^*)^T R K^* = 0$$

$$x^T Q x - x^T (K^*)^T R K^* x$$

$$+\ x^T P A x + x^T A^T P x = 0$$

$$A^T P + P A - (K^*)^T R K^* + Q = 0$$