Lecture 8.

A Class of Computed-Torque Like Control

$$M\ddot{q} + N + \tau_d = \tau \qquad (1)$$

$$\tau = \underline{M}(\ddot{q}_d - \underline{\underline{u}}) + \underline{N} \qquad (2)$$

Computed-torque control

$N = V + G$

Problem: 1. M/N may not be known

2. Computing M/N is slower than action time of robots

If so, we propose

Computed-torque like control

$$\tau_c = \hat{M}(\ddot{q}_d - u) + \hat{N} \qquad (3)$$

Option 1:

PD-Plus-Gravity Control

set
$$\begin{cases} \hat{M} = I, \quad \hat{N} = G(q) - \ddot{q}_d \\ u = -k_v \dot{e} - k_p e \end{cases} \quad (4)$$

Put (4) into (3) to yield

$$\tau_c = I\left(\ddot{q}_d + k_v \dot{e} + k_p e\right) + G(q) - \ddot{q}_d$$

$$= \underline{k_v \dot{e} + k_p e} + \underline{G(q)} \quad (5)$$

We prove τ_c works by proving stability using

Lyapunov Stability

Select $V(x(t))$, if

(1) $V(x) > 0$, iff $x \neq 0$

(2) $V(x) = 0$, iff $x = 0$

(3) $\dot{V}(x) < 0$, for all $x \neq 0$,

Then the dynamics $L(x, \dot{x})$ is stable,

Example,

$$\underline{\ddot{y}(t)} + \underline{\dot{y}(t)} + y(t) = 0$$

To study Lyapunov stability,

we find state-space system by

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases} \Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Select Lyapunov function

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

Then ① and ② conditions hold.

$$\dot{V}(x) = \frac{dV}{dt} = \frac{d(\frac{1}{2}x_1^2)}{dt} + \frac{d(\frac{1}{2}x_2^2)}{dt}$$

$$= x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= x_1 x_2 + x_2(-x_1 - x_2)$$

$$= -x_2^2$$

$$\dot{V}(x) = -x_2^2 < 0, \text{ for } x_2 \neq 0$$

The condition ③ holds; the $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$

is stable; $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to 0$ as time $\to \infty$

To prove τ_c stablizing robot dynamics, we use Lyapunov stability theory by selecting

$$V = \frac{1}{2}(\dot{q}^T M \dot{q} + e^T K_p e)$$

Theorem. Suppose PD-plus-gravity control (5) is used in arm dynamics, and $\tau_d = 0$, $\dot{q}_d = 0$

Then $e = q_d - q \to 0$ as $t \to \infty$.

Proof: $$M\ddot{q} + V(q,\dot{q}) + G(q) = \tau_c \quad (1.1)$$

We have $$\tau_c = K_v\dot{e} + K_p e + G(q) \quad (1.2)$$

Put (1.2) in (1.1) to yield

$$M\ddot{q} + V + \cancel{G(q)} = K_v\dot{e} + K_p e + \cancel{G(q)}$$

Select Lyapunov function V

$$V(\dot{q}, e) = \frac{1}{2}\Big(\underbrace{\dot{q}^T M \dot{q}}_{M>0} + \underbrace{e^T K_p e}_{K_p>0}\Big)$$

① $V(\dot{q}, e) > 0$, iff $\begin{bmatrix} \dot{q} \\ e \end{bmatrix} \neq 0$

② $V(\dot{q}, e) = 0$, iff $\begin{bmatrix} \dot{q} \\ e \end{bmatrix} = 0$

③ $\dot{V}(\dot{q}, e) < 0$,

$$\dot{V} = \frac{1}{2}\Big(\ddot{q}^T M \dot{q} + \dot{q}^T \dot{M} \dot{q} + \dot{q}^T M \ddot{q} + \dot{e}^T K_p e + e^T K_p \dot{e}\Big)$$

$$= \frac{1}{2}\Big(2\ddot{q}^T M \dot{q} + \dot{q}^T \dot{M} \dot{q} + 2\dot{e}^T K_p e\Big)$$

Because of $\dot{e} = \dot{q}_d - \dot{q}$

$$= -\dot{q}$$

Then we have

$$\dot{V} = \frac{1}{2}\left(2\underline{\underline{\ddot{q}^T M}}\dot{q} + \dot{q}^T \dot{M}\dot{q} - 2\dot{q}^T K_p e\right)$$

Recall that

$$\boxed{\begin{aligned} M\ddot{q} + V &= K_v\dot{e} + K_p e \\ (\underline{M\ddot{q}})^T &= (K_v\dot{e} + K_p e - V)^T \end{aligned}}$$

$$\dot{V} = \frac{1}{2}\Big(2\big(\dot{e}^T K_v + \cancel{\underline{e^T K_p}} - V^T\big)\dot{q} + \dot{q}^T \dot{M} \dot{q} - \cancel{\underline{2\dot{q}^T K_p e}} \Big)$$

$$= \dot{e}^T K_v \underline{\underline{\dot{q}}} - V^T \dot{q} + \frac{1}{2}\dot{q}^T \dot{M} \dot{q}$$

$$= -\dot{e}^T K_v \dot{e} - \big(\underbrace{V^T \dot{q} - \tfrac{1}{2}\dot{q}^T \dot{M} \dot{q}}_{=0}\big)$$

$$= -\dot{e}^T K_v \dot{e}$$

$$< 0 \quad \text{iff} \quad \dot{e} \neq 0$$

Classical Joint Control

$$\tau = \hat{M}(\ddot{q}_d - u) + \hat{N}$$

select $\hat{M} = I$

$\hat{N} = -\ddot{q}_d$ → $u - \ddot{q}_d$ in computed-torque like controllers

Then,

$$\tau = I(\ddot{q}_d - u) - \ddot{q}_d$$

$$= -u$$

PD Control:

$$u = -k_v \dot{e} - k_p e$$

$$\tau = k_v \dot{e} + k_p e$$

Simplified Motor Dynamics

$$J\ddot{\theta} + B\dot{\theta} = u - w$$

J, B constant

θ motor angle

u control input

w disturbance

Assume $\ddot{\theta}_d = 0$, $\dot{\theta}_d = 0$, $\theta_d = \hat{\theta}$

$$e = \theta_d - \theta$$

$$\dot{e} = -\dot{\theta}$$

$$\ddot{e} = -\ddot{\theta}$$

Put $\dot{e}, \ddot{e}$ into motor dynamics to yield

$$J(-\ddot{e}) + (-B\dot{e}) = -k_v\dot{e} - k_p e - w$$

$$J\ddot{e} + (B-k_v)\dot{e} - k_p e = w \quad ①$$

Laplace transform (ignoring initial conditions)

$$s^2 J e(s) + (B-k_v) s e(s) - k_p e(s) = w(s)$$

$$(s^2 J + (B-k_v)s - k_p) e(s) = w(s)$$

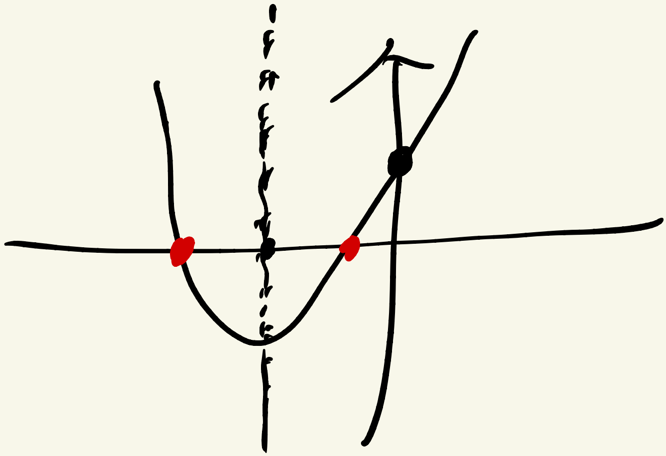
$$\frac{e(s)}{w(s)} = \frac{1}{Js^2 + (B-k_v)s - k_p}$$

$$\Delta(s) = Js^2 + (B-k_v)s - k_p$$

1) The roots of $\Delta(s)$ represents the poles of closed-loop system ①.

2) If all poles are negative, them sysеem is stable, i.e., $e \to 0$

$J > 0$,



$$\begin{cases} -k_p > 0 \\ -\dfrac{B-k_v}{2J} < 0 \end{cases} \quad \to B - k_v > 0$$

$$\Rightarrow \begin{cases} k_p < 0 \\ k_v < B \end{cases} \quad //$$