ELEC 5530/6530
Robotics
Lecture 7: Computed-Torque Control

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$q_d(t)$

The robot arm dynamical equation in Table 3.3.1 is

$$M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q) + \tau_d = \tau, \qquad 1$$

with $q(t) \in R^n$ the joint variable vector $\tau(t)$ and the ~~control input~~ general force. $M(q)$ is the inertia matrix, $V(q, \dot{q})$ the Coriolis/centripetal vector, $F_v(\dot{q})$ the viscous friction, $F_d(\dot{q})$ the dynamic friction, $G(q)$ the gravity, and τ_d a disturbance. These terms satisfy the properties shown in Table 3.3.1. We may also write the dynamics as

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau, \qquad 2$$

with the nonlinear terms represented by

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q).$$

Derivation of Inner Feedforward Loop

Suppose that a desired trajectory $q_d(t)$ has been selected for the arm motion. To ensure trajectory tracking by the joint variable, define an output or *tracking error* as

$$e(t) \triangleq q_d(t) - q(t) \qquad (3)$$

To demonstrate the influence of the input τ on the tracking error, differentiate twice to obtain

$$\dot{e} = \dot{q}_d - \dot{q} \qquad (4)$$

$$\ddot{e} = \ddot{q}_d - \underline{\underline{\ddot{q}}} \qquad (5)$$

Recall that $\ddot{q} = -M^{-1}(N + \tau_d - \tau)$ (6)

Then $\ddot{e} = \ddot{q}_d + M^{-1}(N + \tau_d - \underline{\tau})$ (7) ✓

Let's define control input function

$$u(t) \triangleq \ddot{q}_d + M^{-1}(N - \tau) \quad (8)$$

and the disturbance function

$$w(t) \triangleq M^{-1}\tau_d \quad (9)$$

so,

$$\ddot{e} = u + w \quad (10)$$

(16)

Then, define

$$X = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \quad (11)$$

$$\dot{X} = \begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} \dot{e} \\ u+w \end{bmatrix} \qquad (12)$$

$$= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ I \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

$$\dot{X} = AX + Bu + Dw \qquad (13)$$

From (8), we have this force τ as

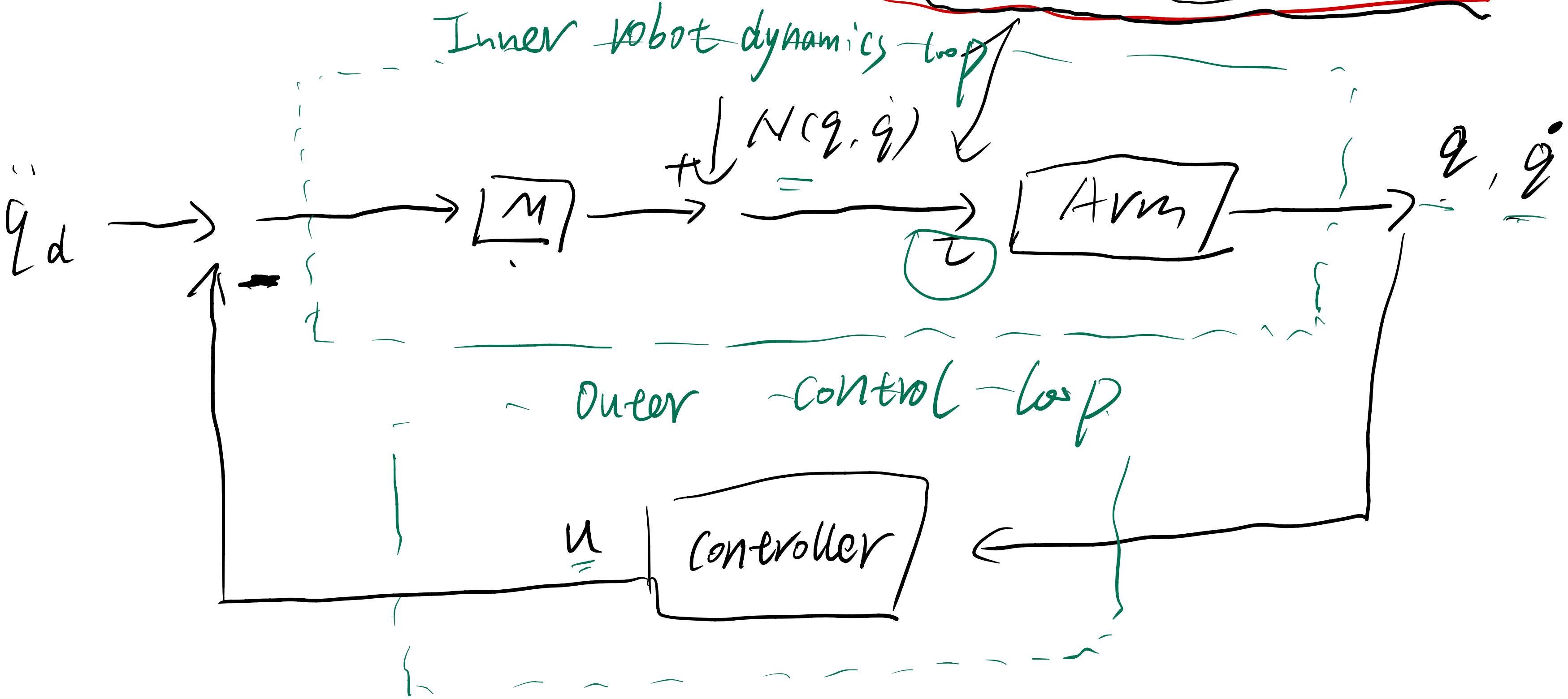
$$\tau = M(\ddot{q}_d - u) + N \qquad (14)$$

$$\Downarrow$$

Computed-torque control

Then we substitute (14) into robot arm dynamics (2)

$$M(q)\ddot{q} + N + \tau_d = M(\ddot{q}_d - u) + N \qquad (15)$$



diagram

% LQR Cont
clear all; clos

Proportional-plus-Derivative (PD) Outer Loop

Let's design $\underline{u} = -k_v \underline{\dot{e}} - k_p \underline{e}$ (16)

k_v: derivative gain

k_p: proportional gain

Putting (16) into (14) yields

$$\tau = \underline{\underline{M}}(\underline{\ddot{q}_d} + k_v \underline{\dot{e}} + k_p \underline{e}) + \underline{\underline{N}} \qquad (17)$$

Putting (17) into (10) yields

$$\ddot{e} = -k_v \dot{e} - k_p e + w \qquad (18)$$

Proportional-plus-Derivative (PD) Outer Loop

$$\ddot{e} + k_v \dot{e} + k_p e - w = 0 \quad (19)$$

w ↑ bounded disturbance

$$\ddot{e} + k_v \dot{e} + k_p e = 0 \quad ✓ \quad (20)$$

State-space representation of (20)

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -k_p & -k_v \end{bmatrix}}_{M} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \quad (21)$$

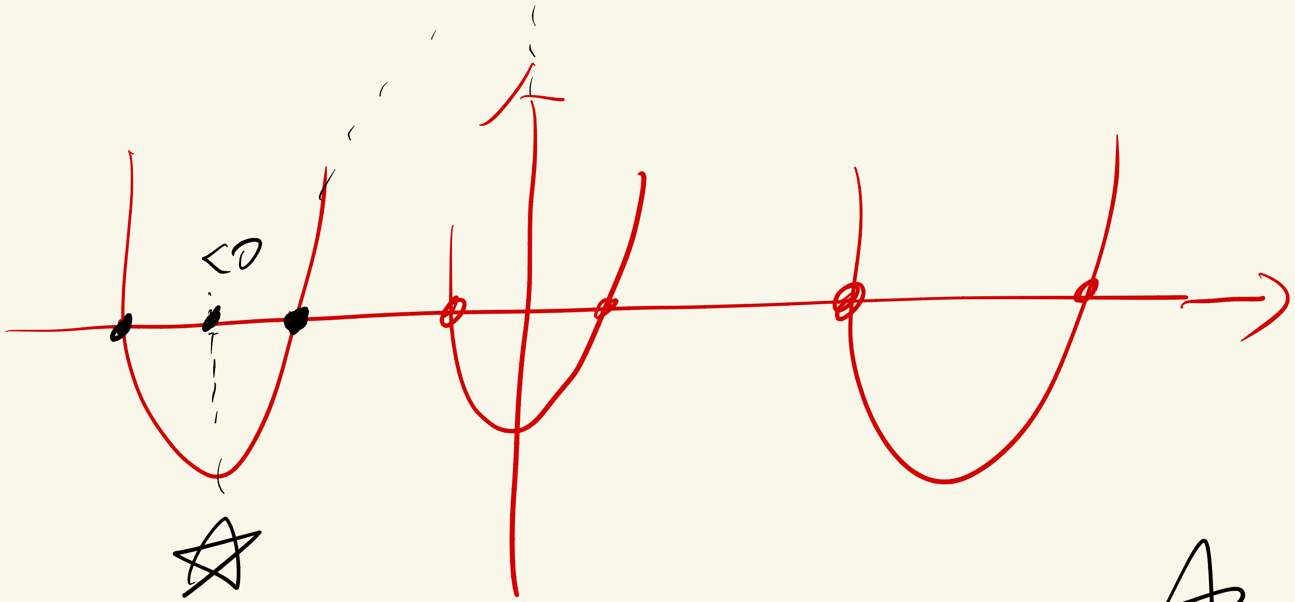
To make (20) asymptotically stable (AS), we need to ensure all eigenvalues of M being negative

$$M = \begin{bmatrix} 0 & I \\ -k_p & -k_v \end{bmatrix}$$

$$|sI - M| = 0 \Rightarrow s^2 I + k_v s + k_p = 0$$

need to have all negative roots s_1, s_2

> 0



If $\begin{cases} -\dfrac{k_v}{2} < 0 \\ k_p > 0 \end{cases}$, then M has negative eigen.

So, (20) is AS, (19) is bounded-input-bounded-output stable.

We just need to selece

$$\begin{cases} K_v > 0 \\ K_p > 0 \end{cases}$$

The easiese selection of them are diagnoal form

$$K_v = \begin{pmatrix} K_{v_1} & 0 & \cdots & 0 \\ 0 & K_{v_2} & & \\ \vdots & & \ddots & \\ 0 & & & K_{v_n} \end{pmatrix} \qquad K_p = \begin{pmatrix} K_{p_1} & 0 & \cdots & 0 \\ 0 & K_{p_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & & K_{p_n} \end{pmatrix}$$

Part b: PID Control

proportional-integral-derivative

Robot Arm Motion Equation

$$M\ddot{q} + N + \tau_d = \tau$$

Tracking Error $e = q_d(t) - q(t)$

$$\dot{e} = \dot{q}_d - \dot{q}$$

$$\ddot{e} = \ddot{q}_d - \ddot{q}$$

$$= \ddot{q}_d - M^{-1}(\tau - N - \tau_d)$$

$$= \ddot{q}_d + M^{-1}(N + \tau_d - \tau)$$

Define control inpue function

$$u \triangleq \ddot{q}_d + M^{-1}(N - \tau)$$

disturbance input function

$$\omega \triangleq M^{-1} \tau_d$$

Then, $\ddot{e} = u + \omega$

Design PID controller

feedback on state e

$$\begin{cases} u = -k_v \dot{e} - k_p e - k_I \varepsilon \\ \dot{\varepsilon} = e \quad (\varepsilon = \int e) \end{cases}$$

k_v : derivative gain

?

k_P : proportional gain
k_I : integral gain

to be designed

put the PID u into $\ddot{e}$

$$\ddot{e} = -k_v \dot{e} - k_P e - k_I \varepsilon + w$$

$$\dot{e} = \dot{e}$$

$$\dot{\varepsilon} = e$$

$\downarrow$ state-space representation

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -k_I & -k_p & -k_v \end{bmatrix} \begin{bmatrix} \varepsilon \\ e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} w$$

$$\dot{X} = A X + B \underbrace{w}_{\text{bounded}}$$

Now we need to make sure $\dot{X} = AX$ is stable

Then we can have bounded syseem $\dot{X} = AX + Bw$

(Bw is bounded)

To get stable A, we need to have all negative eigenvalues of A.

$$|sI - A| = \begin{vmatrix} sI & -I & 0 \\ 0 & sI & -I \\ k_I & k_p & sI + k_v \end{vmatrix} = 0$$

$$= |s^3 I + k_v s^2 + k_p s + k_I| = 0$$

$$\underline{s^3 I + k_v s^2 + k_p s + k_I = 0}$$

Set k_v, k_p, k_I as diagnol matrices

$$k_v = \begin{bmatrix} k_{v_1} & & & \\ & k_{v_2} & & \\ & & \ddots & \\ & & & k_{v_n} \end{bmatrix} \qquad k_p = \begin{bmatrix} k_{p_1} & & & \\ & k_{p_2} & & \\ & & \ddots & \\ & & & k_{p_n} \end{bmatrix}$$

$$k_I = \begin{bmatrix} k_{I_1} & & & \\ & k_{I_2} & & \\ & & \ddots & \\ & & & k_{I_n} \end{bmatrix}_{n \times n}$$

That means $s^3 + k_{v_i} s^2 + k_{p_i} s + k_{I_i} = 0, \quad i \in (1, 2, \cdots n)$

Rowth Test

$$p(s) = s^3 + k_{v_i} s^2 + k_{p_i} s + k_{I_i} + 0 \text{ Any}$$

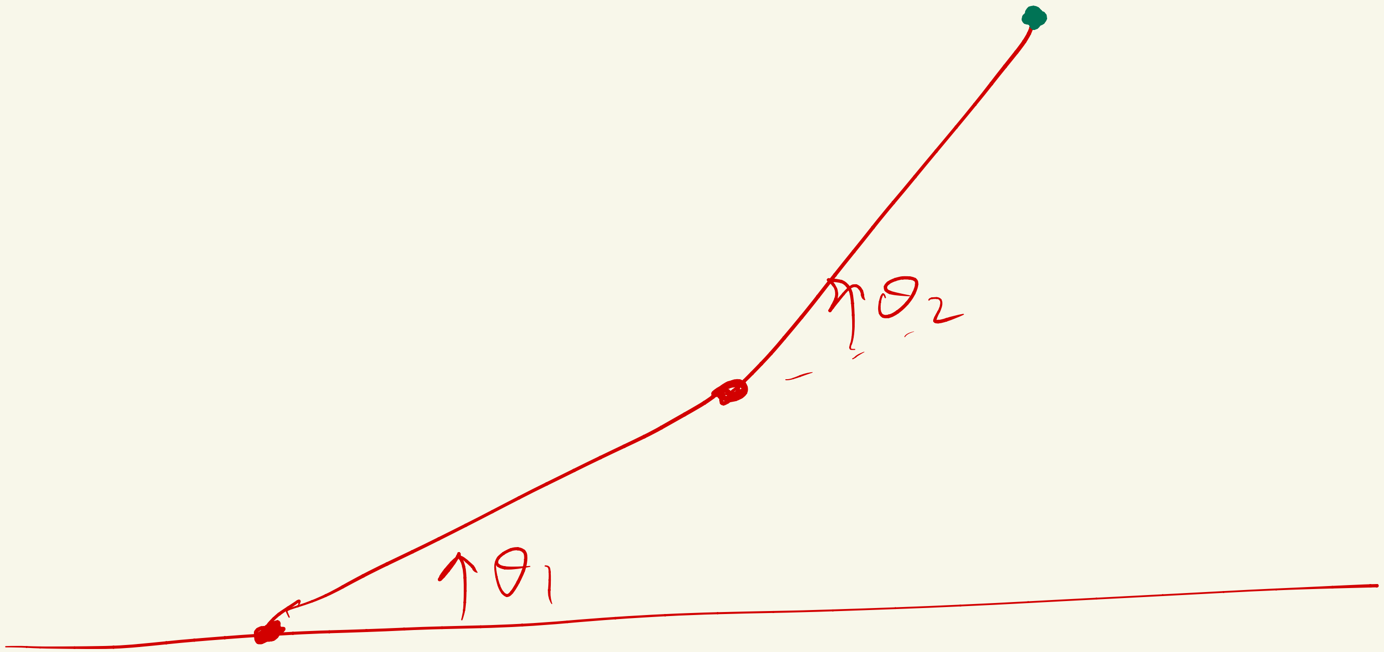
$$
\begin{array}{llll}
 & \text{Column 1} & & \\
s^3 & \underline{1} & k_{p_i} & 0 \\
s^2 & k_{v_i} & k_{I_i} & 0 \\
s & \boxed{} & & \\
s^0 & k_{I_i} & &
\end{array}
$$

$$-\begin{vmatrix} 1 & k_{p_i} \\ k_{v_i} & k_{I_i} \end{vmatrix} \Big/ k_{v_i}$$

$$= \frac{k_{p_i} k_{v_i} - k_{I_i}}{k_{v_i}}$$

don't want sign changes in column 1

$$\begin{cases} 1 > 0 \\ K_{V_i} > 0 \\ \dfrac{K_{P_i} K_{V_i} - K_{I_i}}{K_{V_i}} > 0 \\ K_{I_i} > 0 \end{cases} \Rightarrow \begin{cases} K_{V_i} > 0 \\ K_{I_i} > 0 \\ K_{P_i} > 0 \end{cases}$$



$\theta_1 \to \theta_{1d} \to$ desired trajectory

$\theta_2 \to \theta_{2d} \to$ - - - -

$$\underbrace{[(m_1+m_2)a_1^2+m_2a_2^2+2m_2a_1a_2\cos\theta_2]}_{\triangleq A}\ddot{\theta}_1 \quad \underbrace{+\,m_2 g a_2\cos(\theta_1+\theta_2) + (m_1+m_2)g a_1\cos\theta_1}_{\triangleq g_1}$$

$$+\underbrace{[m_2a_2^2+m_2a_1a_2\cos\theta_2]}_{\triangleq B}\ddot{\theta}_2 - \underbrace{m_2a_1a_2(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^2)\sin\theta_2}_{\triangleq v_1} = \tau_1$$

$$\underbrace{[m_2a_2^2+m_2a_1a_2\cos\theta_2]}_{\triangleq C}\ddot{\theta}_1 + \underbrace{m_2a_2^2}_{\triangleq D}\ddot{\theta}_2 + \underbrace{m_2a_1a_2\dot{\theta}_1^2}_{\triangleq v_2}$$

$$+\underbrace{m_2 g a_2\cos(\theta_1+\theta_2)}_{\triangleq g_2} = \tau_2$$

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\ddot{q}} + \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{V} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}}_{G} = \underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}}_{\tau}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} + \begin{bmatrix} \tau_{d1} \\ \tau_{d2} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Recall that PD control

$$e(t) = q_d(t) - q(t) \qquad \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad \begin{bmatrix} \theta_{1d} \\ \theta_{2d} \end{bmatrix}$$

$$\dot{e}(t) = \dot{q}_d - \dot{q} \qquad - \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_{1d} \\ \dot{\theta}_{2d} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\ddot{e}(t) = \ddot{q}_d - \ddot{q}$$

$$u = -k_v \dot{e} - k_p e \rightarrow \begin{bmatrix} k_{p_1} & 0 \\ 0 & k_{p_2} \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\tau = M(\ddot{q}_d - u) + N$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \left(\begin{pmatrix} \ddot{\theta}_{1d} \\ \ddot{\theta}_{2d} \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) + \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \left[\begin{array}{l} \dot{q} = \dot{q} \\ \ddot{q} = M^{-1}(\tau - N - \tau_d) \end{array} \right.$$