ELEC 5530/6530
Robotics
Lecture 6: Robot Dynamics: State-Variable Representations and Feedback Linearization

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The robot arm dynamical equation in Table 3.3.1 is

$$M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q) + \tau_d = \tau,$$

with $q(t) \in R^n$ the joint variable vector $\tau(t)$ and the control input. $M(q)$ is the inertia matrix, $V(q, \dot{\boldsymbol{q}})$ the Coriolis/centripetal vector, $F_v(\dot{\boldsymbol{q}})$ the viscous friction, $F_d(\dot{\boldsymbol{q}})$ the dynamic friction, $G(q)$ the gravity, and τ_d a disturbance. These terms satisfy the properties shown in Table 3.3.1. We may also write the dynamics as

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau,$$

with the nonlinear terms represented by

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q).$$

Some equivalent formulations of the arm dynamical equation.

1. The nonlinear state-variable representation

$$\dot{x} = f(x, u, t) = f(x) + g(x) \cdot u$$

2. The linear state-variable representation

$$\frac{dx}{dt} \leftarrow \underbrace{\dot{x} = Ax + Bu. + Dw}_{}$$

disturbance

Hamiltonian Formulation

The arm equation was derived using Lagrangian mechanics. Here, let us use Hamiltonian mechanics [Marion 1965] to derive a state-variable formulation of the manipulator dynamics [Arimoto and Miyazaki 1984], [Gu and Loh 1985]. Let us neglect the friction terms $F(\dot{q}) = F_v(\dot{q}) + F_d(\dot{q})$ and the disturbance τ_d for simplicity; they may easily be added at the end of our development.

we expressed the arm Lagrangian as

$$L = K - P = \tfrac{1}{2}\dot{q}^T M(q)\,\dot{q} - P(q)$$

①

with $q(t) \in R^n$ the joint variable, K the kinetic energy, P the potential energy, and $M(q)$ the arm inertia matrix. Define the *generalized momentum* by

$$K = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 = \frac{1}{2}\begin{bmatrix}\dot{\theta} & \dot{r}\end{bmatrix}\underbrace{\begin{bmatrix} mr^2 & 0 \\ 0 & m\end{bmatrix}}_{2\times 2}\begin{bmatrix}\dot{\theta} \\ \dot{r}\end{bmatrix}$$

$$\dot{q} = \begin{bmatrix}\dot{\theta} \\ \dot{r}\end{bmatrix} \qquad = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

this is for 2-link polar arm

Define the generalized momentum by

$$p \triangleq \frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1} \\ \frac{\partial L}{\partial \dot{q}_2} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_n} \end{bmatrix} = M(q)\dot{q} \qquad (2)$$

Then, from (2), $\dot{q} = M^{-1} p \qquad (3)$

$$K = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$= \frac{1}{2} p^T M^{-1} M M^{-1} p = \frac{1}{2} p^T \underline{M^{-1} p} \quad (4)$$

$$K = \frac{1}{2} p^T \dot{q} \quad (5)$$

Define Hamiltonian by

$$\begin{cases} H = p^T \dot{q} - L & (6.1) \\ \dot{q} = \frac{\partial H}{\partial p} & (6.2) \end{cases}$$

$$-\dot{p} = \frac{\partial H}{\partial q} - \tau \qquad (6.3)$$

For (6.1)

$$H = 2K - (K - \bar{P}), \text{ where } \bar{P} \text{ is potential energy}$$

$$= K + \bar{P} \qquad (7)$$

For (6.3)

$$-\dot{p} = \frac{\partial H}{\partial q} - \tau$$

$$= \frac{\partial \left(\frac{1}{2} p^T M^{-1} p + \bar{P}(q) \right)}{\partial q} - \tau$$

$$= \frac{1}{2}\underbrace{(I_n \otimes P)}_{\text{to make dimension match}} \frac{\partial M}{\partial q} P + G(q) - \tau$$

Then $$\underline{\underline{\dot{P}}} = -\frac{1}{2}(I_n \otimes \underline{P}) \frac{\partial M}{\partial q} \underline{P} - G(q) + \tau$$

$$\dot{q} = M^{-1} P$$

Define $X = \begin{bmatrix} P \\ q \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} -\frac{1}{2}(I_n \otimes x_1)\frac{\partial M}{\partial x_2} x_1 - G(x_2) + \tau \\ M^{-1} x_1 \end{bmatrix}$$

$$= f(x) + g(x) \cdot u$$

$$f(x) \triangleq \begin{bmatrix} -\frac{1}{2}(I_n \otimes x_1)\frac{\partial M}{\partial x_2} x_1 - G(x_2) \\ M^{-1} x_1 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Position/Velocity Formulations in joint space

Alternative state-space formulations of the arm dynamics may be obtained by defining the position/velocity state $x \in R^{2n}$ as

e.g. three-link model

$$x \triangleq \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in R^{2n}$$

$$q = \begin{bmatrix} \theta \\ h \\ r \end{bmatrix} \in R^3$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}$$

Recall that $M\ddot{q} + N + \tau_d = \tau$

$$\ddot{q} = -M^{-1}N - M^{-1}\tau_d + M^{-1}\tau$$

where $N = V(q,\dot{q}) + F_v\dot{q} + F_d + G(q)$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ -M^{-1}N - M^{-1}\tau_d + M^{-1}\tau \end{bmatrix} \checkmark$$

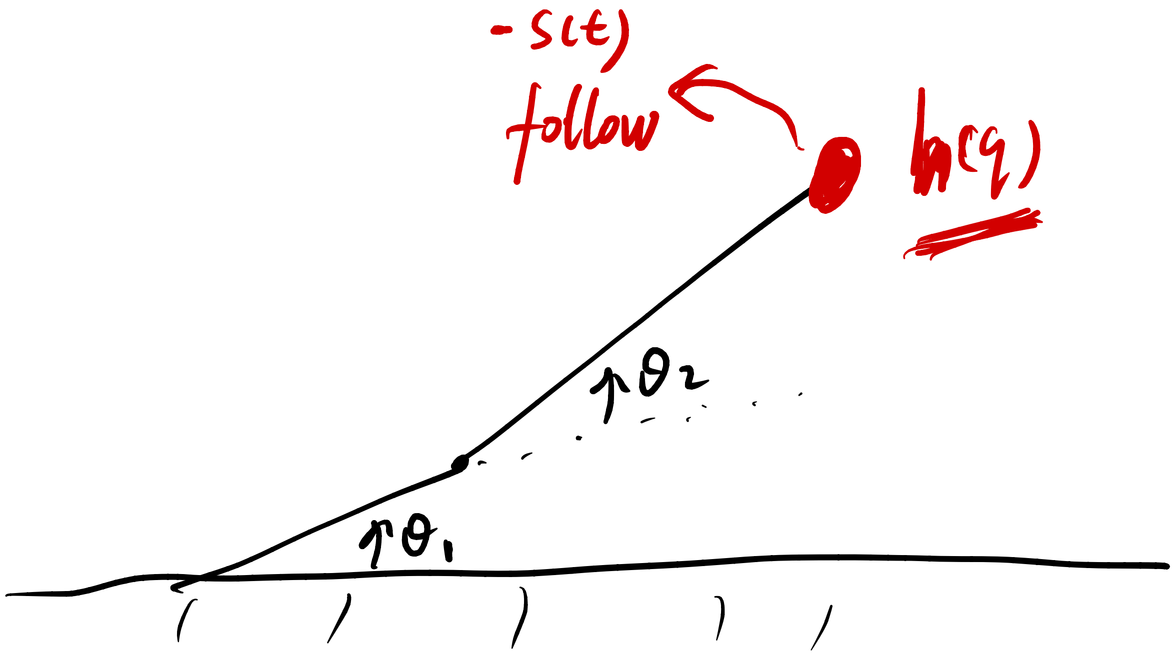
$$= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} w \quad \checkmark$$

☆ $\underline{u} \triangleq \underline{-M^{-1}N + M^{-1}\tau} \longrightarrow$ control input

$\underline{w} \triangleq \underline{-M^{-1}\tau_d} \longrightarrow$ disturbance

$$\dot{x} = Ax + Bu + Dw$$

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ I \end{bmatrix}$$



$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Feedback Linearization

Let us now develop a general approach to the determination of linear state-space representations of the arm dynamics. The technique involves a linearization transformation that removes the manipulator nonlinearities. It is a simplified version of the feedback linearization technique [Hunt et al. 1983, Gilbert and Ha 1984].

Let us define a general sort of output by $= h(q) - (-s(t))$

$$y = h(q) + s(t),$$

with $h(q)$ a general predetermined function of the joint variable $q \in R^n$ and $s(t)$ a general predetermined time function. The control problem, then, will be to select the joint torque and force inputs $\tau(t)$ in order to make the output $y(t)$ go to zero.

translation vector

$\begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

The selection of $h(q)$ and $s(t)$ is based on the control objectives we have in mind. For instance, if $h(q)=-q$ and $s(t)=q_d(t)$, the desired joint space trajectory we would like the arm to follow, then $y(t)=q_d(t)-q(t)=e(t)$ the joint space tracking error. Forcing $y(t)$ to zero in this case would cause the joint variables $q(t)$ to track their desired values $q_d(t)$, resulting in arm trajectory following.

$$y(t) = h(q) + s(t)$$

$$\dot{y}(t) = \frac{d\,h(q)}{dt} + \frac{d\,s(t)}{dt}$$

$$= \frac{\partial h}{\partial q} \cdot \frac{\partial q(t)}{\partial t} + \dot{s}$$

$$= J \cdot \dot{q} + \dot{s}$$

$$\ddot{y} = \frac{d(J \cdot \dot{q})}{dt} + \ddot{s}$$

$$= \dot{J}\dot{q} + J\ddot{q} + \ddot{s}$$

so, $\ddot{y} = \dot{J}\dot{q} + J\ddot{q} + \ddot{s}$

$$\boxed{\text{Recall that} \quad M\ddot{q} + N + \tau_d = \tau}$$

$$\boxed{\ddot{q} = -M^{-1}N - M^{-1}\tau_d + M^{-1}\tau}$$

$$\ddot{y} = \underline{\dot{J}\dot{q} - JM^{-1}N + JM^{-1}\tau}\ \underline{-JM^{-1}}\tau_d + \underline{\ddot{s}}$$

Let's define

$$u \triangleq J\dot{q} - JM^{-1}N + JM^{-1}\tau + \ddot{s}$$

$$w \triangleq -JM^{-1}\tau_d$$

Then $\ddot{y} = u + w$

$$z_1 \triangleq y$$

$$z_2 \triangleq \dot{y}$$

$$\dot{z}_1 = \dot{y}$$

$$\dot{z}_2 = \ddot{y} = u + w$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

$$\dot{z} = A z + B u + D w$$

Cartesian Dynamics

$$M(q)\ddot{q} + \underline{V(q,\dot{q}) + G(q)} + \tau_d = \tau$$

or

$$M\ddot{q} + \underline{N(q,\dot{q})} + \tau_d = \tau \quad (1)$$

$$y = h(q)$$

y is the Cartesian or task space position of end effector

$$\dot{y} = \frac{\partial h}{\partial q} \cdot \frac{\partial q}{\partial t}$$

$$= J \cdot \dot{q}$$

$$\ddot{y}_{6\times1} = \frac{d(J \cdot \dot{q})}{dt}$$

$$= \dot{J}\dot{q} + J_{6\times6}\,\ddot{q}_{6\times1}$$

The cartesian velocity $\dot{y} = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$, so $J \in \mathbb{R}^{6\times6}$

$$\ddot{q} = \underline{J^{-1}\ddot{y} - J^{-1}\dot{J}\dot{q}} \quad (2)$$

Replace $\ddot{q}$ in (1) with (2) to yield

$$M(J^{-1}\ddot{y} - J^{-1}\dot{J}\dot{q}) + N + \tau_d = \underline{\underline{\tau}}$$

force for angle — $\tau = J^T F$

$$M J^{-1} \ddot{y} + N - M J^{-1} \dot{J} \dot{q} + \tau_d = \underset{6\times 6}{J^T} \underline{\underline{F}}$$

$$J^{-T} M J^{-1} \ddot{y} + J^{-T}(N - M J^{-1} \dot{J} \dot{q}) + J^{-T} \tau_d = F$$

$$\bar{M} \ddot{y} + \underline{\bar{N}} + \underline{\bar{\tau}_d} = F \rightarrow \text{Cartesian dynamics}$$

vs

$$M \underline{\ddot{q}} + N + \tau_d = \tau \rightarrow \text{joint dynamics}$$