ELEC 5530/6530
Robot Control
Lecture 5: Robot Dynamics: Structure and Properties of the Robot Equation

Instructor: Dr. Bosen Lian
Electrical and Computer Engineering
Auburn University
Spring 2026

Motivation:

The controller is simpler and more effective if the known properties of the arm are incorporated in the design stage.

Robot Arm Model with *friction* and *disturbances*

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) + \tau_d = \tau,$$

with q the joint variable n-vector and τ the n-vector of generalized forces. $M(q)$ is the inertia matrix, $V(q, \dot{q})$ the Coriolis/centripetal vector, and $G(q)$ the gravity vector. We have added a *friction* term

$$F(\dot{q}) = F_v\dot{q} + F_d$$

with F_v the coefficient matrix of *viscous friction* and F_d a *dynamic friction* term. Also added is a *disturbance* τ_d, which could represent, for instance, any inaccurately modeled dynamics.

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau,$$

where

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F(\dot{q}) + G(q)$$

represents nonlinear terms.

TABLE 2.3-1 The Robot Equation and Its Properties

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) + \tau_d = \tau$$

or

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau$$

where

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F(\dot{q}) + G(q)$$

one norm

$\| \cdot \|_1$

Inertia Matrix:

$M(q)$ is symmetric and positive definite.

$$\mu_1 I \le M(q) \le \mu_2 I$$

$$m_1 \le \|M(q)\| \le m_2$$

Coriolis/Centripetal Vector:

$V(q,\dot{q})$ is quadratic in $\dot{q}$

$$\|V(q,\dot{q})\| \le v_b \|\dot{q}\|^2$$

$$V(q,\dot{q}) = V_m(q,\dot{q})\dot{q}$$

$S(q,\dot{q}) \equiv \dot{M}(q) - 2V_m(q,\dot{q})$ is a skew-symmetric matrix.

Friction Terms:

$$F(\dot{q}) = F_v\dot{q} + F_d(\dot{q})$$

$$F_v = \text{diag}\{v_i\}$$

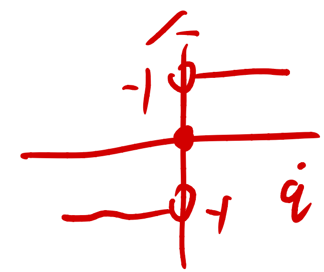
$$\begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & & \\ \vdots & & \ddots & \\ 0 & & & v_n \end{bmatrix}$$

$$F_d(\dot{q}) = K_d \,\text{sgn}(\dot{q}), \text{ with } K_d = \text{diag}\{k_i\}$$

$$\|F_v\dot{q} + F_d(\dot{q})\| \le \mathbf{v}\|\dot{q}\| + \mathbf{k}$$

$$\begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{bmatrix}$$

$\text{sgn}(\dot{q})$



Gravity Vector:

$$\|G(q)\| \le g_b$$

Disturbance Term:

$$\|\tau_d\| \le \mathbf{d}$$

$$\text{sgn}(\dot{q}) = \begin{cases} -1, & \text{if } \dot{q} < 0 \\ 0, & \text{if } \dot{q} = 0 \\ 1, & \text{if } \dot{q} > 0 \end{cases}$$

Linearity in the Parameters:

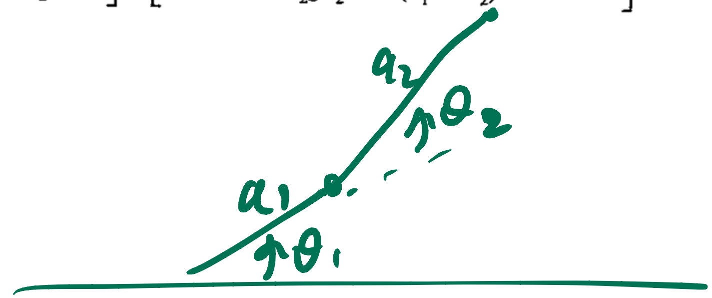
$$M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q)$$

$$= M(q)\ddot{q} + N(q,\dot{q}) \equiv W(q,\dot{q},\ddot{q})\varphi$$

Example 1. Structure and Bounds for Two-Link Planar Arm Model

$$\begin{bmatrix} (m_1 + m_2)\, a_1^2 + m_2 a_2^2 + 2m_2 a_1 a_2 \cos\theta_2 & m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2 \\ m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2 & m_2 a_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -m_2 a_1 a_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \sin\theta_2 \\ m_2 a_1 a_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} + \begin{bmatrix} (m_1 + m_2)\, g a_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1 + \theta_2) \\ m_2 g a_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$= \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$



These manipulator dynamics are in the standard form

$$\underline{M(q)}\ddot{q} + \underline{V(q,\dot{q})} + \underline{G(q)} = \tau,$$

$$M(q) = \begin{bmatrix} \overset{M_{11}}{\underline{(m_1 + m_2)a_1^2 + m_2 a_2^2 + 2m_2 a_1 a_2 \cos\theta_2}} & \overset{M_{12}}{\underline{m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2}} \\ \underset{M_{21}}{\underline{m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2}} & \underset{M_{22}}{\underline{m2 a_2^2}} \end{bmatrix}$$

$$V(q,\dot{q}) = \begin{bmatrix} -m_2 a_1 a_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \sin\dot{\theta}_2 \\ m_2 a_1 a_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} \begin{matrix} \to V_1 \\ \to V_2 \end{matrix}$$

$$G(q) = \begin{bmatrix} (m_1 + m_2) g a_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1 + \theta_2) \\ m_2 g a_2 \cos(\theta_1 + \theta_2) \end{bmatrix}.$$

1 norm of vector $\|\cdot\|_1 \rightarrow$ one norm

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

exp.

$$x = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \qquad \|x\|_1 = 1 + 0 + 2 = 3$$

1 norm of matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \triangleq [x_{ij}]$$

$$\|X\|_1 \triangleq \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$$

$$= \max\{ |x_{11}| + |x_{21}| + \cdots + |x_{n1}|$$

$$\vdots$$

$$|x_{1n}| + |x_{2n}| + \cdots + |x_{nn}| \}$$

$$X = \begin{bmatrix} -3 & -5 & 7 \\ 2 & 6 & 4 \\ 0 & 2 & 8 \end{bmatrix}, \quad \|X\|_1 = \max\{ \begin{matrix} |-3| + |2| + 0 \\ |-5| + |6| + |2| \\ |7| + |4| + |8| \end{matrix} \}$$

a. Bounds on the Intertia Matrix $= 19$

$$\|M(q)\|_1 = \max\{|M_{11}| + |M_{21}|,\ |M_{12}| + |M_{22}|\}$$

Due to $|a+b| \le |a| + |b|$,

$$|M_{11}| \le (m_1+m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2 \triangleq a$$

$$|M_{21}| \le m_2a_2^2 + m_2a_1a_2 \triangleq b$$

$$|m_2a_2^2 - m_2a_1a_2| + m_2a_2^2 \le \|M(q)\|_1 \le a+b$$

b. Bounds on the Centripetal Term

$$\|V\|_1 = |V_1| + |V_2|$$

$$|V_1| \le m_2 a_1 a_2 \left(|2\dot{\theta}_1 \dot{\theta}_2| + \dot{\theta}_2^2 \right)$$

$$|V_2| \le m_2 a_1 a_2 \dot{\theta}_1^2$$

$$|V_1| + |V_2| \le m_2 a_1 a_2 \left(\underline{\dot{\theta}_1^2 + 2|\dot{\theta}_1 \dot{\theta}_2|} + \dot{\theta}_2^2 \right)$$

$$\le m_2 a_1 a_2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2$$

c. Bounds on the Friction Term

$$F(\dot{q}) = F_v \dot{q} + F_d$$

$$F_v = \mathrm{diag}[v_i] \rightarrow \begin{bmatrix} |v_1| & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_n \end{bmatrix}$$

$$F_d = K_d \,\mathrm{sgn}(\dot{q}), \quad K_d = \mathrm{diag}[k_i]$$

$$\|F(\dot{q})\|_1 \le \underline{\|F_v \dot{q}\|_1} + \underline{\|F_d\|_1}$$

$$\le |v_{max}||\dot{q}| + |K_{max}|$$

d. Bounds on the Gravity and Disturbance Terms

$$G(q) = \begin{bmatrix} (m_1+m_2) g a_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1+\theta_2) \\ m_2 g a_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\|G(q)\|_1 = \left|(m_1+m_2) g a_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1+\theta_2)\right| + \left|m_2 g a_2 \cos(\theta_1+\theta_2)\right|$$

$$\leq \left|(m_1+m_2) g a_1 \cos\theta_1\right| + \left|m_2 g a_2 \cos(\theta_1+\theta_2)\right|$$

$$\left|m_2 g a_2 \cos(\theta_1+\theta_2)\right| \leq (m_1+m_2) g a_1 + m_2 g a_2 + m_2 g a_2$$

e. The Robot Function Parameter Matrix W

$$\tau_1 = [(m_1 + m_2)a_1^2 + m_2 a_2^2 + 2m_2 a_1 a_2 \cos\theta_2]\ddot{\theta}_1 + [m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2]\ddot{\theta}_2 - m_2 a_1 a_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 + (m_1 + m_2) g a_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1 + \theta_2) + \underline{v_1\dot{\theta}_1} + k_1 \underline{\operatorname{sgn}(\dot{\theta}_1)}$$

$$\tau_2 = [m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2]\ddot{\theta}_1 + m_2 a_2^2 \ddot{\theta}_2 + m_2 a_1 a_2 \dot{\theta}_1^2 \sin\theta_2 + m_2 g a_2 \cos(\theta_1 + \theta_2) + v_2\dot{\theta}_2 + k_2 \operatorname{sgn}(\dot{\theta}_2).$$

$$\tau_{2\times 1} = W_{2\times 6}\,\varphi_{6\times 1}, \quad \text{where } \varphi = \begin{bmatrix} m_1 \\ m_2 \\ k_1 \\ k_2 \\ v_1 \\ v_2 \end{bmatrix}_{6\times 1}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{16} \\ w_{21} & w_{22} & w_{23} & w_{24} & w_{25} & w_{26} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ k_1 \\ k_2 \\ v_1 \\ v_2 \end{bmatrix}_{6\times 1}$$

$$w_{11} = a_1^2\ddot{\theta}_1 + \theta a_1 \cos\theta_1$$

$$w_{12} = (a_2^2 + a_1^2 + 2a_1 a_2 \cos\theta_2)\ddot{\theta}_1 + (a_2^2 + a_1 a_2 \cos\theta_2)\ddot{\theta}_2$$

$$W_{13} = \mathrm{sgn}(\dot{\theta}_1)$$

$$W_{14} = 0$$

$$W_{15} = \dot{\theta}_1$$

$$W_{16} = 0$$

$$W_{21} = 0$$

$$W_{22} = a_2^2\ddot{\theta}_1 + a_1 a_2 \cos\theta_2 \ddot{\theta}_1 + a_2^2\ddot{\theta}_2 + a_1 a_2 \dot{\theta}_1^2 \sin\theta_2 + g a_2 \cos(\theta_1+\theta_2)$$

$$W_{23} = 0$$

$$W_{24} = \mathrm{sgn}(\dot{\theta}_2)$$

$$W_{25} = 0$$

$$W_{26} = \dot{\theta}_2$$