Control of Robot Manipulators

F. L. Lewis
University of Texas at Arlington

C. T. Abdallah
University of New Mexico

D. M. Dawson
Clemson University

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2.3 Structure and Properties of the Robot Equation

In this section we investigate the detailed structure and properties of the dynamical arm equations, for this structure should be reflected in the form of the control law. The controller is simpler and more effective if the known properties of the arm are incorporated in the design stage.

In reality, a robot arm is always affected by friction and disturbances. Therefore, we shall generalize the arm model we have just derived by writing the manipulator dynamics as

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) + \tau_d = \tau, \tag{2.3-1}$$

with q the joint variable n-vector and τ the n-vector of generalized forces. $M(q)$ is the inertia matrix, $V(q,\dot{q})$ the Coriolis/centripetal vector, and $G(q)$ the gravity vector. We have added a *friction* term

$$F(\dot{q}) = F_v\dot{q} + F_d \tag{2.3-2}$$

with F_v the coefficient matrix of *viscous friction* and F_d a *dynamic friction* term. Also added is a *disturbance* τ_d, which could represent, for instance, any inaccurately modeled dynamics.

Friction is not an easy term to model, and indeed, may be the most contrary term to describe in the manipulator dynamics model. Some more discussion on friction may be found in [Schilling 1990].

We shall sometimes write the arm dynamics as

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau, \tag{2.3-3}$$

where

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F(\dot{q}) + G(q) \tag{2.3-4}$$

represents nonlinear terms.

Let us examine the structure and properties of each of the terms in the robot dynamics equation. This study will offer us a great deal of insight which we use in deriving robot control schemes in subsequent chapters. A summary of the properties we discover is given in Table 2.3-1, to which we refer in the remainder of the book. As we develop each property, it will be worthwhile to refer to Examples 2.2-1 to 2.2-3 in order to verify that the properties indeed hold there. At the end of this section we illustrate in Example 2.3-1 several of the properties for a two-link planar elbow arm.

TABLE 2.3-1 The Robot Equation and Its Properties

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) + \tau_d = \tau$$

or

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau$$

where

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F(\dot{q}) + G(q)$$

Inertia Matrix:

$M(q)$ is symmetric and positive definite.

$$\mu_1 I \leq M(q) \leq \mu_2 I$$

$$m_1 \leq \|M(q)\| \leq m_2$$

Coriolis/Centripetal Vector:

$V(q,\dot{q})$ is quadratic in $\dot{q}$

$$\|V(q,\dot{q})\| \leq v_b \|\dot{q}\|^2$$

$$V(q,\dot{q}) = V_m(q,\dot{q})\dot{q}$$

$S(q,\dot{q}) \equiv \dot{M}(q) - 2V_m(q,\dot{q})$ is a skew-symmetric matrix.

Friction Terms:

$$F(\dot{q}) = F_v\dot{q} + F_d(\dot{q})$$

$$F_v = \mathrm{diag}\{v_i\}$$

$F_d(\dot{q}) = K_d \,\mathrm{sgn}(\dot{q})$, with $K_d = \mathrm{diag}\{k_i\}$

$$\|F_v\dot{q} + F_d(\dot{q})\| \leq \mathbf{v}\|\dot{q}\| + \mathbf{k}$$

Gravity Vector:

$$\|G(q)\| \leq g_b$$

Disturbance Term:

$$\|\tau_d\| \leq \mathbf{d}$$

Linearity in the Parameters:

$$M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q)$$

$$= M(q)\ddot{q} + N(q,\dot{q}) \equiv W(q,\dot{q},\ddot{q})\varphi$$

EXAMPLE 2.3-1: Structure and Bounds for Two-Link Planar Elbow Arm

The dynamics of a two-link planar arm are given in Example 2.2-2. We should now like to compute the structural matrices defined in this section, as well as the bounds needed in Table 2.3-1. The friction bounds are straightforward, so we do not mention them here. The dynamical matrices are

$$M(q) = \begin{bmatrix} (m_1 + m_2)a_1^2 + m_2 a_2^2 + 2m_2 a_1 a_2 \cos\theta_2 & m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2 \\ m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2 & m2a_2^2 \end{bmatrix}$$

$$V(q,\dot{q}) = \begin{bmatrix} -m_2 a_1 a_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\dot{\theta}_2 \\ m_2 a_1 a_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} (m_1 + m_2)ga_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1 + \theta_2) \\ m_2 g a_2 \cos(\theta_1 + \theta_2) \end{bmatrix}.$$

The selection of a suitable norm in Table 2.3-1 is not always straightforward. In the control algorithms to be developed in subsequent chapters, we prove suitable performance in terms of some norm, which can often be any norm desired. For implementation of the controller, a specific norm must be selected and the bounds evaluated. This choice often depends simply on which norm makes it possible to evaluate the bounds in the table. For instance, choosing the 2-norm for vectors requires the evaluation of the maximum singular value of $M(q)$, a very difficult task.

Selecting the ∞-norm for vectors means determining at each sampling time the element [of $V(q(t),\dot{q}(t))$ for instance] with the largest magnitude. This requires decision logic, and the norm may not be continuous. Therefore, let us use the 1-norm in this example. The corresponding matrix induced norm is then the maximum absolute column sum (Chapter 1).

a. Bounds on the Inertia Matrix

The evaluation of μ_1 and μ_2 amounts to the determination of the minimum and maximum eigenvalues of $M(q)$ over all q. This is not an easy affair and requires the solution of some quadratic equations, although it can be carried out without too much trouble using software such as Mathematica or Maple. Thus, let us find m_1 and m_2.

The induced 1-norm for $M(q)$ is the maximum absolute column sum. In determining bounds for this norm, it is important to consider the range of allowed motion of the joint angles. To illustrate, suppose that θ_1 and θ_2 are limited by $\pm\pi/2$. Then the 1-norm is always given in terms of column 1 as

$$\|M(q)\|_1 = |(m_1 + m_2)a_1^2 + m_2 a_2^2 + 2m_2 a_1 a_2 \cos\theta_2| + |m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2|,$$

which is bounded above for all θ_2 by

$$M_2 = (m_1 + m_2)a_1^2 + 2m_2a_2^2 + 3m_2a_1a_2$$

and below by

$$M_1 = (m_1 + m_2)a_1^2 + 2m_2a_2^2.$$

Since the arm is revolute and $\cos\theta_2$ is bounded above and below, M_2 and M_1 are constants. It is important to note that if the arm is revolute/prismatic (RP), so that the joint variables are (θ_1, a_2), the bounds are functions of q.

b. Bounds on the Coriolis and Gravity Terms

The bound v_b on the Coriolis/centripetal vector is found using

$$\begin{aligned}\|V(q,\dot{q})\|_1 &= |m_2a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2| + |m_2a_1a_2\dot{\theta}_1^2\sin\theta_2| \\ &\leq m_2a_1a_2(|\dot{\theta}_1| + |\dot{\theta}_2|)^2 \equiv v_b\|\dot{q}\|^2,\end{aligned}$$

whence $v_b = m_2a_1a_2$.

Similarly, for the gravity bound,

$$\begin{aligned}\|G(q)\|_1 &= |(m_1 + m_2)ga_1\cos\theta_1 + m_2ga_2\cos(\theta_1 + \theta_2)| + |m_2ga_2\cos(\theta_1 + \theta_2)| \\ &\leq (m_1 + m_2)ga_1 + 2m_2ga_2 \equiv g_b.\end{aligned}$$

Notice that if the arm is RP, then v_b and g_b are functions of q.

c. Coriolis/Centripetal Structural Matrices

We now list the various structural matrices for $V(q,\dot{q})$ discussed in this section. Their computation is left as an exercise (see the Problems).

$$U = (I \otimes \dot{q}^T)\frac{\partial M}{\partial q} = \begin{bmatrix} 0 & 0 \\ -(2\dot{\theta}_1 + \dot{\theta}_2)m_2a_1a_2\sin\theta_2 & -\dot{\theta}_1m_2a_1a_2\sin\theta_2 \end{bmatrix}$$

The Coriolis/centripetal matrices:

$$V_{m1} = \dot{M} - \tfrac{1}{2}U = \begin{bmatrix} -2\dot{\theta}_2m_2a_1a_2\sin\theta_2 & -\dot{\theta}_2m_2a_1a_2\sin\theta_2 \\ (\dot{\theta}_1 - \tfrac{1}{2}\dot{\theta}_2)m_2a_1a_2\sin\theta_2 & \tfrac{1}{2}\dot{\theta}_1m_2a_1a_2\sin\theta_2 \end{bmatrix}$$

$$V_{m2} = U^T - \tfrac{1}{2}U = \begin{bmatrix} 0 & -(2\dot{\theta}_1 + \dot{\theta}_2)m_2a_1a_2\sin\theta_2 \\ (\dot{\theta}_1 + \tfrac{1}{2}\dot{\theta}_2)m_2a_1a_2\sin\theta_2 & \tfrac{1}{2}\dot{\theta}_1m_2a_1a_2\sin\theta_2 \end{bmatrix}$$

$$V_m = \tfrac{1}{2}(\dot{M} + U^T - U) = \begin{bmatrix} -\dot{\theta}_2m_2a_1a_2\sin\theta_2 & -(\dot{\theta}_1 + \dot{\theta}_2)m_2a_1a_2\sin\theta_2 \\ \dot{\theta}_1m_2a_1a_2\sin\theta_2 & 0 \end{bmatrix}$$

The skew-symmetric matrix $S(q,\dot{q})$:

$$S(q,\dot{q}) = U - U^T = \begin{bmatrix} 0 & (2\dot{\theta}_1 + \dot{\theta}_2)m_2a_1a_2 \sin\theta_2 \\ -(2\dot{\theta}_1 + \dot{\theta}_2)m_2a_1a_2 \sin\theta_2 & 0 \end{bmatrix}$$

The symmetric matrices $V_1(q,\dot{q})$, $V_2(q,\dot{q})$:

$$V_1 = \begin{bmatrix} 0 & -m_2a_1a_2 \sin\theta_2 \\ -m_2a_1a_2 \sin\theta_2 & -m_2a_1a_2 \sin\theta_2 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} m_2a_1a_2 \sin\theta_2 & 0 \\ 0 & 0 \end{bmatrix}$$

The position/velocity decomposition matrices:

$$V_p(q) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2m_2a_1a_2 \sin\theta_2 & -m_2a_1a_2 \sin\theta_2 \\ -m_2a_1a_2 \sin\theta_2 & 0 \end{bmatrix}$$

$$V_v(\dot{q}) = \begin{bmatrix} \tfrac{1}{2}\dot{\theta}_1 & -\tfrac{1}{2}\dot{\theta}_2 & \dot{\theta}_2 & 0 \\ 0 & \dot{\theta}_1 & -\tfrac{1}{2}\dot{\theta}_1 & \tfrac{1}{2}\dot{\theta}_2 \end{bmatrix}$$

d. The Robot Function Parameter Matrix W

The robot dynamics are linear in the parameters. For the purposes of adaptive control, one should select the parameter vector φ in Table 2.3-1 so that it contains the unknown parameters.

The dynamics, including friction, can be written as

$$\begin{aligned}\tau_1 = {} & [(m_1 + m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2 \cos\theta_2]\ddot{\theta}_1 + [m_2a_2^2 + m_2a_1a_2 \cos\theta_2]\ddot{\theta}_2 \\ & - m_2a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \sin\theta_2 + (m_1 + m_2)ga_1 \cos\theta_1 + m_2ga_2 \cos(\theta_1 + \theta_2) \\ & + v_1\dot{\theta}_1 + k_1 \operatorname{sgn}(\dot{\theta}_1)\end{aligned}$$

$$\begin{aligned}\tau_2 = {} & [m_2a_2^2 + m_2a_1a_2 \cos\theta_2]\ddot{\theta}_1 + m_2a_2^2\ddot{\theta}_2 + m_2a_1a_2\, \dot{\theta}_1^2 \sin\theta_2 \\ & + m_2ga_2 \cos(\theta_1 + \theta_2) + v_2\dot{\theta}_2 + k_2 \operatorname{sgn}(\dot{\theta}_2).\end{aligned}$$

The second mass m_2 includes the mass of the payload. This and the friction coefficients are often unknown. Therefore, select

$$\varphi = [m_1 \quad m_2 \quad k_1 \quad v_1 \quad k_2 \quad v_2]^T.$$

Then the matrix $W(q,\dot{q},\ddot{q})$ of known robot functions becomes

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & 0 & 0 \\ 0 & w_{22} & 0 & 0 & w_{25} & w_{26} \end{bmatrix}$$

with

$$\begin{aligned}
w_{11} &= a_1^2\ddot{\theta}_1 + ga_1\cos\theta_1 \\
w_{12} &= [a_1^2 + a_2^2 + 2a_1a_2\cos\theta_2]\,\ddot{\theta}_1 + [a_2^2 + a_1a_2\cos\theta_2]\ddot{\theta}_2 \\
&\quad - a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 + ga_1\cos\theta_1 + ga_2\cos(\theta_1 + \theta_2) \\
w_{13} &= \mathrm{sgn}(\dot{\theta}_1) \\
w_{14} &= \dot{\theta}_1 \\
w_{22} &= [a_2^2 + a_1a_2\cos\theta_2]\,\ddot{\theta}_1 + a_2^2\ddot{\theta}_2 + a_1a_2\dot{\theta}_1^2\sin\theta_2 + ga_2\cos(\theta_1 + \theta_2) \\
w_{25} &= \mathrm{sgn}(\dot{\theta}_2) \\
w_{26} &= \dot{\theta}_2 .
\end{aligned}$$

The reader should verify that with these definitions, the dynamics may be expressed as $\tau = W\varphi$. The matrix W is computed from measured joint positions, and their velocities and accelerations.

Passivity and Conservation of Energy

The "Newtonian" form of the manipulator dynamics given in Table 2.3-1 obscures some important physical properties, which we should like to explore here [Koditschek 1984, Slotine 1988, Slotine and Li 1987, Ortega and Spong 1988, Johansson 1990]. Note that the dynamics can be written in terms of the skew-symmetric matrix $S(q,\dot{q})$ as

$$M(q)\ddot{q} + \tfrac{1}{2}(\dot{M}(q) - S(q,\dot{q}))\dot{q} = \tau - G(q), \tag{2.3-63}$$

where friction and τ_d are ignored. Now, with K the kinetic energy, we have

$$\frac{dK}{dt} = \frac{1}{2}\frac{d}{dt}\dot{q}^T M(q)\dot{q} = \dot{q}^T M\ddot{q} + \tfrac{1}{2}\dot{q}^T\dot{M}\dot{q},$$

whence (2.3-63) yields

$$\dot{K} = \tfrac{1}{2}\dot{q}^T S\dot{q} + \dot{q}^T(\tau - G) \tag{2.3-64}$$

or

$$\dot{K} = \dot{q}^T(\tau - G). \tag{2.3-65}$$

This is a statement of the conservation of energy, with the right-hand side representing the power input from the net external forces. The skew symmetry of $S = (\dot{M} - 2V_m)$ is nothing more than a statement that the fictitious forces $S(q,\dot{q})\dot{q}$ do no work. The work done by the external forces is given by

$$K = \int \dot{q}^T(\tau - G)\,dt. \tag{2.3-66}$$