ELEC 5530/6530Robotics
Lecture 4: Robot Dynamics: Lagrange-Euler Dynamics

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- For **control design** purposes, it is necessary to have a **mathematical model** that reveals the **dynamical behavior** of a system.
- Therefore, in this section we derive the **dynamical equations of motion** for a robot manipulator.
- Our approach is to derive the **kinetic and potential energy** of the manipulator and then use **Lagrange's equations of motion**.

Force, Inertia, and Energy

The *centripetal force* of a mass m orbiting a point at a radius r and angular velocity ω is given by

$$F_{\text{cent}} = \frac{mv^2}{r} = m\omega^2 r = m\dot{\theta}^2 r.$$

$\dot{\theta} = \omega$

The linear velocity is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$,

which in this case means simply that $v = \omega r$.

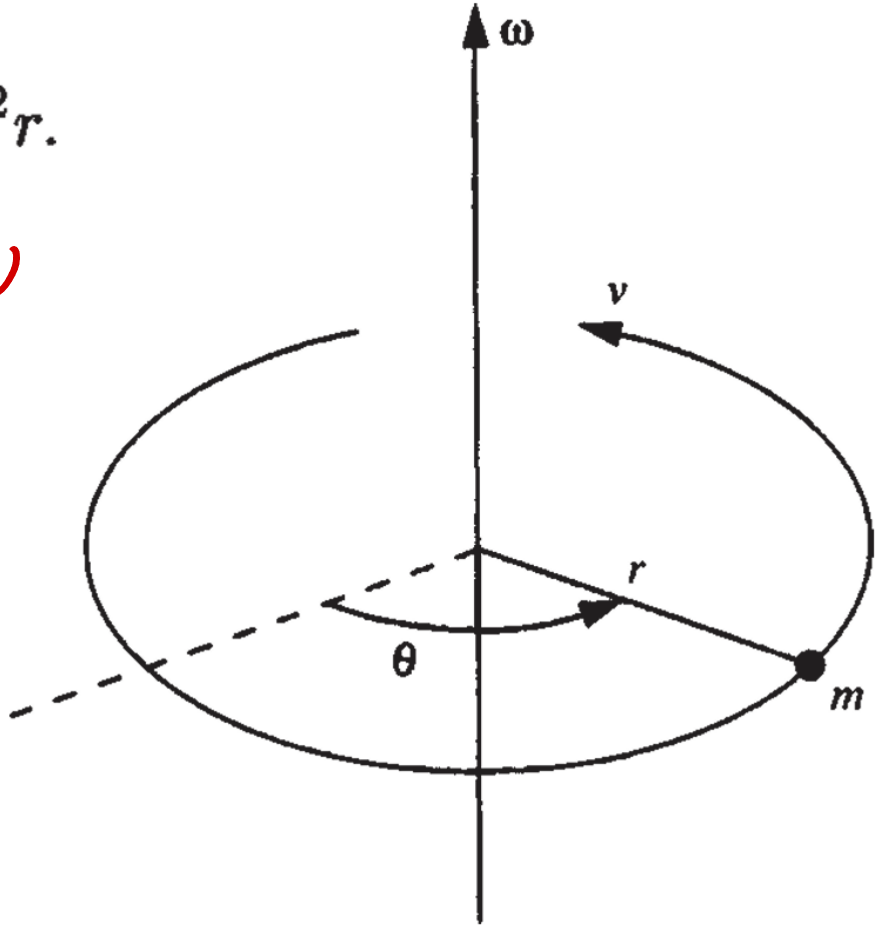


Figure 1. Centripetal force

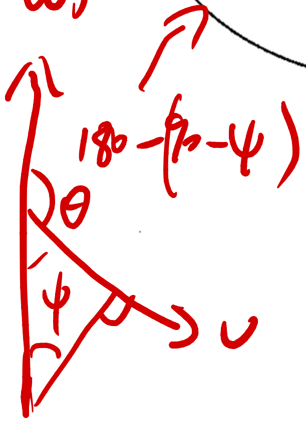
Imagine a sphere (i.e., the earth) rotating about its center with an angular velocity of $\boldsymbol{\omega_0}$. See . The *Coriolis force* on a body of mass m moving with velocity v on the surface of the sphere is given by

$$\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega}_0 \times \mathbf{v}.$$

Using the right-handed screw rule (i.e., if the fingers rotate $\boldsymbol{\omega_0}$ into $\boldsymbol{v}$, the thumb points in the direction of $\boldsymbol{\omega_0} \times \mathbf{v}$, we see that, in the figure, the Coriolis force acts to deflect m to the right.

$\theta = 90+\phi$

ω_0



Since $\omega_0 = \dot\theta$ *and* $v = R\dot\varphi$, *we may write*

$$F_{\text{cor}} = -2m\dot\theta\dot\varphi R\sin(90^\circ + \varphi) = -2mR\dot\theta\dot\varphi\cos\varphi.$$

$$\|\omega_0\|\,\|v\|\cos\theta = \|\omega_0\|\,\|v\|\cos(90+\phi)$$

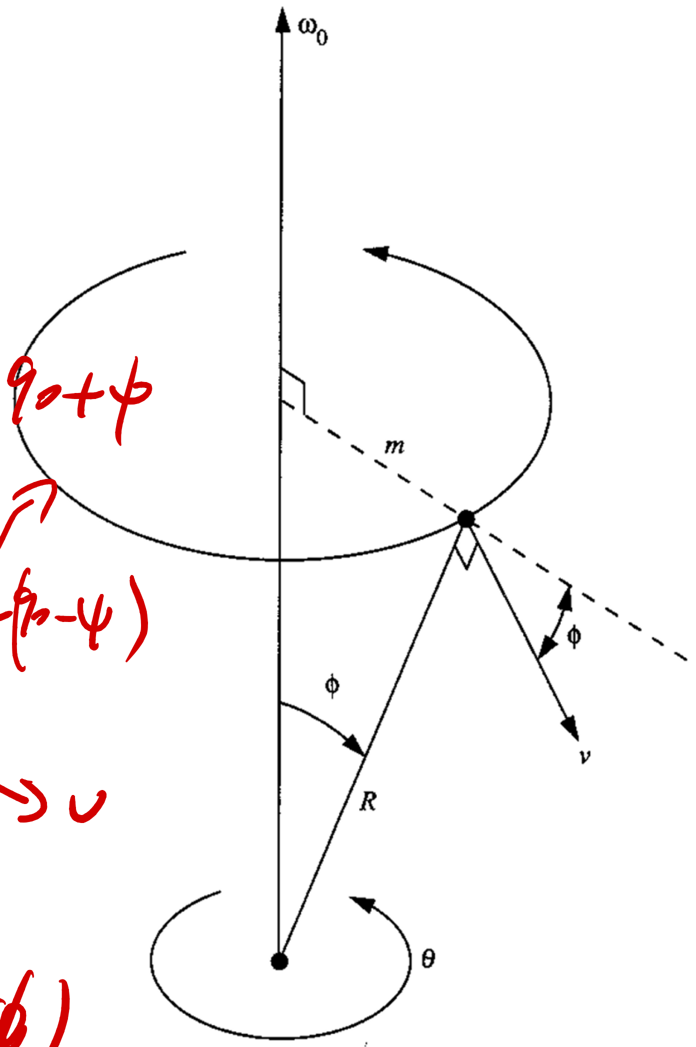


Figure 2. Coriolis force

- The centripetal force involves the square of a single angular velocity,
- The Coriolis force involves the product of two distinct angular velocities.

m

The *kinetic energy* of a mass moving with a linear velocity of v is

$$K = \frac{1}{2}mv^2.$$

The *rotational kinetic energy* of the mass in Figure 1 is given by

$$K_{\text{rot}} = \frac{1}{2}\mathbf{I}\omega^2,$$

where the *moment of inertia* is

$$I = \int_{\text{vol}} \rho(r)r^2\,dr,$$

with $\rho(r)$ the mass distribution at radius r in a volume. In the simple case shown where m is a point mass, this becomes

$$I=mr^2.$$

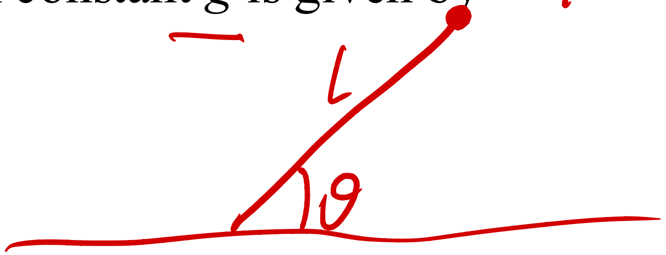
Therefore,

$$K_{\text{rot}} = \frac{1}{2}mr^2\dot{\theta}^2.$$

The *potential energy* of a mass m at a height h in a gravitational field with constant g is given by

$P = mgl\sin\theta$

$$P = mgh.$$



The *momentum* of a mass m moving with velocity v is given by

$$\mathbf{p} = m\mathbf{v}.$$

The *angular momentum* of a mass m with respect to an origin from which the mass has distance r is

$$\mathbf{P}_{\mathrm{ang}} = \mathbf{r} \times \mathbf{p}.$$

The *torque or moment* of a force $\mathbf{F}$ with respect to the same origin is defined to be

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}.$$

1. $L = K - P$

Lagrange's Equations of Motion

Lagrange's equation of motion for a conservative system are given by [Marion 1965]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau,$$

2. $\frac{\partial L}{\partial \dot{q}}$

where q is an n-vector of generalized coordinates q_i, τ is an n-vector of generalized forces τ_i, and the Lagrangian is the difference between the kinetic and potential energies

L

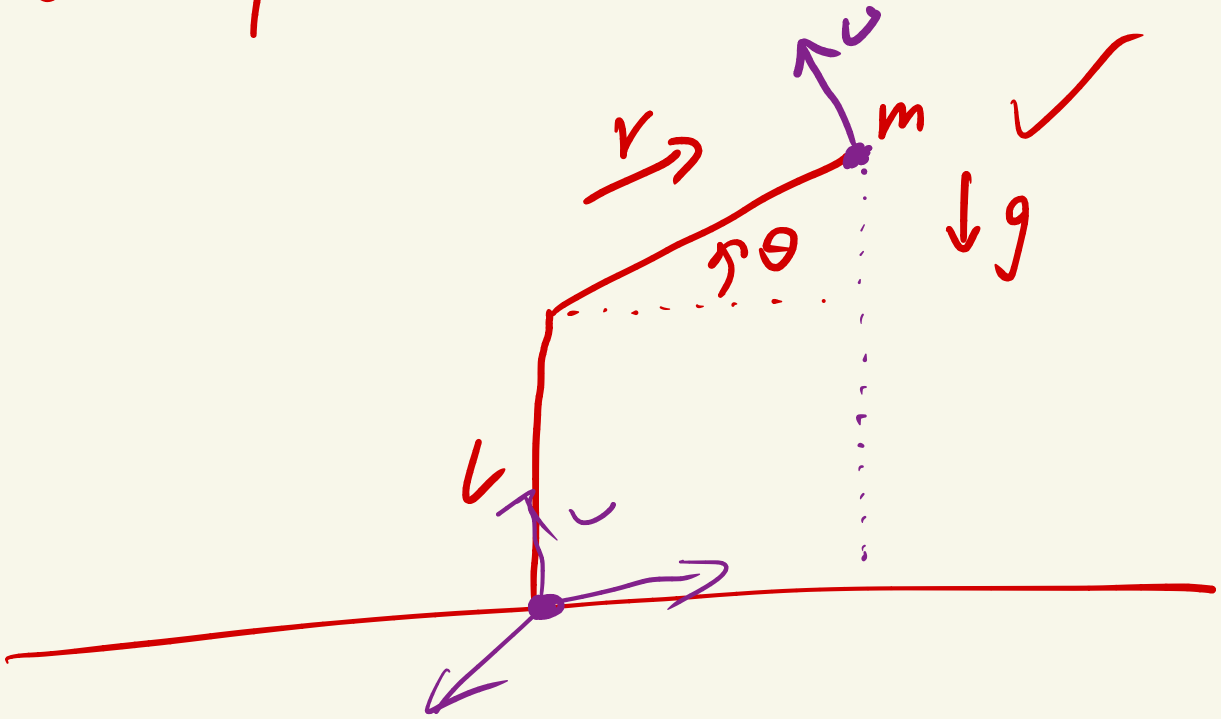
3. $\frac{d \frac{\partial L}{\partial \dot{q}}}{dt}$

$$L = K - P.$$

In our usage, q will be the joint-variable vector, consisting of joint angles θ_i; (in degrees or radians) and joint offsets d_i (in meters). Then τ is a vector that has components n_i of torque (newton-meters) corresponding to the joint angles, and f_i of force (newtons) corresponding to the joint offsets. Note that we denote the scalar components of τ by lowercase letters.

4. $\frac{\partial L}{\partial q}$

Example 1. two-link polar arm



Find equation of motion of end-effector with mass m.

Step 1: $L = K - P$

potential energy

$$P = mg(L + r\sin\theta)$$

$$K = \frac{1}{2} m v^2$$ v is linear velocity

translation vector of mass m

$$p = \begin{bmatrix} r\cos\theta \\ r\sin\theta + L \\ 0 \end{bmatrix}$$

$$v^2 = \left(\frac{d(r\cos\theta)}{dt}\right)^2 + \left(\frac{d(r\sin\theta + L)}{dt}\right)^2 + 0$$

For a coordinate $y = \begin{bmatrix} y_x \\ y_y \\ y_z \end{bmatrix}$

it has velocity $\dot{y} = \begin{bmatrix} \dot{y}_x \\ \dot{y}_y \\ \dot{y}_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$

linear velocity is $v^2 = v_x^2 + v_y^2 + v_z^2$

$$\left(\frac{d\,(r(t)\cos\theta(t))}{dt}\right)^2 = \left(\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta}\right)^2$$

$$= \dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2$$

$$- 2\dot{r}r\cos\theta\sin\theta\,\dot{\theta}$$

$$\left(\frac{d(r\sin\theta + l)}{dt}\right)^2 = \left(\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta}\right)^2$$

$$= \dot{r}^2\sin^2\theta + r^2\cos^2\theta\,\dot{\theta}^2$$

$$+ 2\dot{r}\sin\theta\, r\cos\theta\,\dot{\theta}$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

so, $K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$$L = K - P = \underline{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mg(l + r\sin\theta)}$$

Step 2: $\dfrac{\partial L}{\partial \dot{q}}$

$$\dot{q} = \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{\theta}} & \frac{\partial L}{\partial \dot{r}} \end{bmatrix}^T$$

$$= \begin{bmatrix} mr^2\dot{\theta} & m\dot{r} \end{bmatrix}^T$$

Step 3 :

$$\frac{d \frac{\partial L}{\partial \dot{q}}}{dt} = \begin{bmatrix} 2mr\dot{\theta}\dot{r} + mr^2\ddot{\theta} \\ \\ m\ddot{r} \end{bmatrix}$$

Step 4 :

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial \theta} \\ \\ \frac{\partial L}{\partial r} \end{bmatrix} = \begin{bmatrix} -mgr\cos\theta \\ \\ mr\dot{\theta}^2 - mg\sin\theta \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

$$
\begin{cases}
m\ddot{\theta} r^2 + 2m\theta r\dot{r} + mgr\cos\theta = n \\[2em]
m\ddot{r} - m\dot{\theta}^2 r + mg\sin\theta = f
\end{cases}
$$

$$\Downarrow$$

$$
\begin{bmatrix} mr^2 & 0 \\ 0 & m \end{bmatrix}
\begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix}
+
\begin{bmatrix} 2m\theta r\dot{r} \\ -m\dot{\theta}^2 r \end{bmatrix}
+
\begin{bmatrix} mgr\cos\theta \\ mg\sin\theta \end{bmatrix}
=
\begin{bmatrix} n \\ f \end{bmatrix}
$$

$$
M(q)\,\ddot{q} + V(q,\dot{q}) + G(q) = f \quad \bigstar
$$

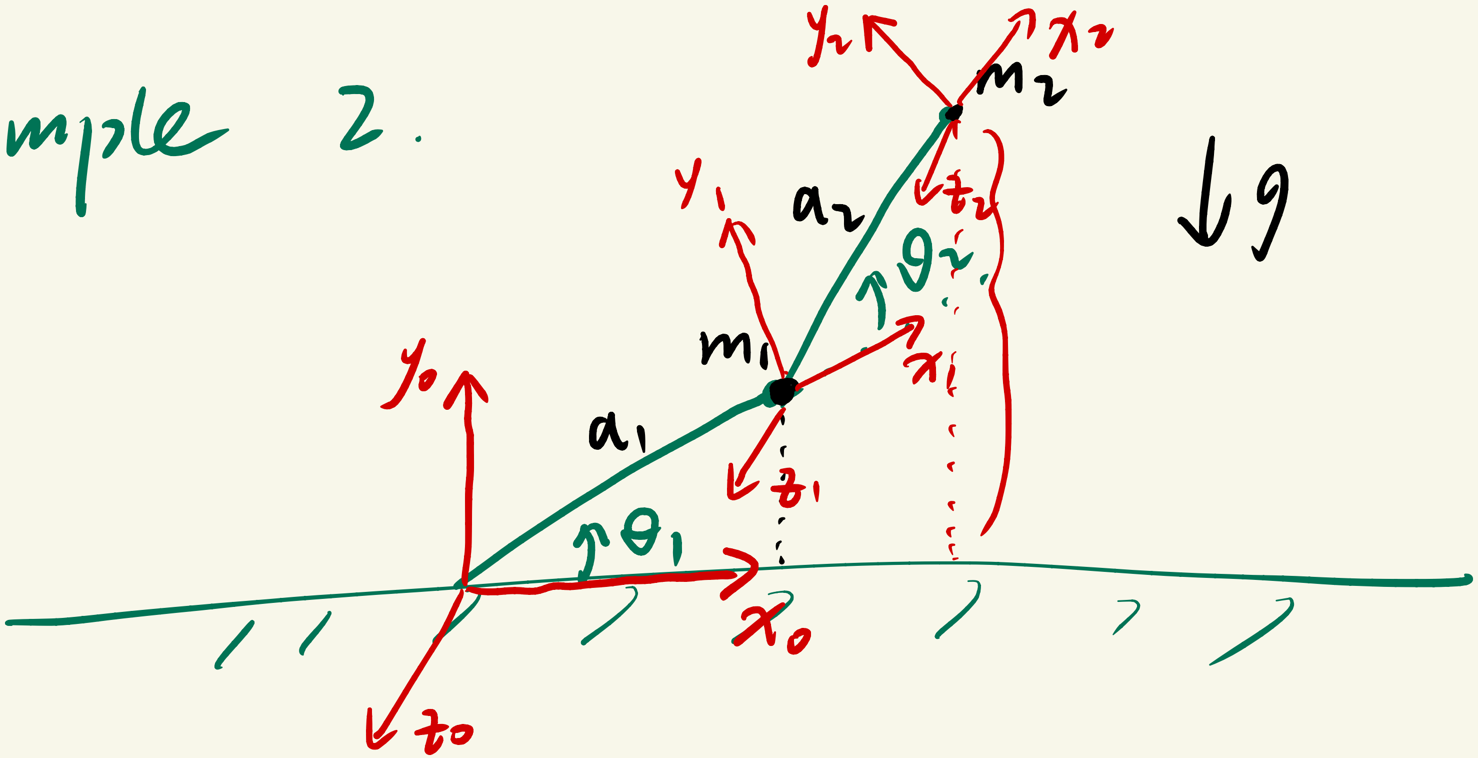
inertria matrix

centripetal vector

gravity vector

generalized force

Example 2.



seep 1 : $L = L_1 + L_2$

$$q = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix}$$

$$L_1 = K_1 - P_1$$

$$L_2 = K_2 - P_2$$

$$K_1 = \frac{1}{2} m_1 v_1^2$$

for m_1, we have coordinate $\begin{bmatrix} a_1 \cos\theta_1 \\ a_1 \sin\theta_1 \end{bmatrix}$

$$v_1^2 = \left(\frac{d(a_1 \cos\theta_1)}{dt}\right)^2 + \left(\frac{d(a_1 \sin\theta_1)}{dt}\right)^2$$

$$= (-a_1 \sin\theta_1 \cdot \dot{\theta}_1)^2 + (a_1 \cos\theta_1 \cdot \dot{\theta}_1)^2$$

$$= a_1^2 \dot{\theta}_1^2$$

$$P_1 = m_1 g a_1 \sin\theta_1$$

$$L_2 = \frac{1}{2} m_2 v_2^2 ,$$

The coordinatee of m_2 is $\begin{pmatrix} P_x = a_1 \cos\theta_1 + a_2 \cos(\theta_1+\theta_2) \\ \\ P_y = a_1 \sin\theta_1 + a_2 \sin(\theta_1+\theta_2) \end{pmatrix}$

$$v_2^2 = \dot{P}_x^2 + \dot{P}_y^2$$

$$= \left(-a_1 \sin\theta_1 \dot{\theta}_1 - a_2 (\dot{\theta}_1+\dot{\theta}_2) \sin(\theta_1+\theta_2)\right)^2$$

$$\frac{d\cos\theta}{dt} = -\sin\theta \cdot \dot{\theta}$$

$$+ \left(a_1 \cos\theta_1 \, \dot{\theta}_1 + a_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \right)^2$$

$$= \underline{a_1^2 \sin^2\theta_1 \, \dot{\theta}_1^2} + \underline{a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \sin^2(\theta_1 + \theta_2)}$$

$$+ \underline{2 a_1 a_2 \sin\theta_1 \sin(\theta_1 + \theta_2) \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2)}$$

$$+ \underline{a_1^2 \cos^2\theta_1 \, \dot{\theta}_1^2} + \underline{a_2^2 \cos^2(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)^2}$$

$$+ \underline{2 a_1 \cos\theta_1 \, \dot{\theta}_1 \, a_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)}$$

$$= a_1^2 \dot{\theta}_1^2 + a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$+ 2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos\theta_2$$

$$P_2 = m_2 g\left(a_1 \sin\theta_1 + a_2 \sin(\theta_1+\theta_2)\right)$$

$$L_2 = K_2 - P_2$$

$$= \frac{1}{2} m_2 a_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos\theta_2$$

$$- m_2 g\left(a_1 \sin\theta_1 + a_2 \sin(\theta_1+\theta_2)\right)$$

The Lagrangian for the entire arm is

$$L = L_1 + L_2$$

$$= \frac{1}{2}(m_1+m_2)a_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 a_2^2(\dot{\theta}_1+\dot{\theta}_2)^2$$

$$+ m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)\cos\theta_2$$

$$-(m_1+m_2)g a_1 \sin\theta_1 - m_2 g a_2 \sin(\theta_1+\theta_2)$$

Step 2. $\frac{\partial L}{\partial \dot{q}}$: $\frac{\partial L}{\partial \dot{\theta}_1}$ $\frac{\partial L}{\partial \dot{\theta}_2}$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)a_1^2\dot{\theta}_1 + m_2 a_2(\dot{\theta}_1+\dot{\theta}_2) + m_2 a_1 a_2(2\dot{\theta}_1+\dot{\theta}_2)\cos\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 a_2^2(\dot{\theta}_1+\dot{\theta}_2) + m_2 a_1 a_2 \dot{\theta}_1 \cos\theta_2$$

Step 3.

$$\frac{d \frac{\partial L}{\partial \dot{\theta}_1}}{dt} = (m_1+m_2) a_1^2 \ddot{\theta}_1 + m_2 a_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 a_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) \cos\theta_2 - m_2 a_1 a_2 (2\dot{\theta}_1 + \dot{\theta}_2)(\sin\theta_2)\dot{\theta}_2 \qquad 1$$

$$\frac{d \frac{\partial L}{\partial \dot{\theta}_2}}{dt} = m_2 a_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 a_2 \ddot{\theta}_1 \cos\theta_2 - m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 \qquad 3$$

step 4. $\frac{\partial L}{\partial q}$: $\frac{\partial L}{\partial \theta_1}$; $\frac{\partial L}{\partial \theta_2}$

$$\frac{\partial L}{\partial \theta_1} = (m_1+m_2)ga_1\cos\theta_1 - m_2 g a_2 \cos(\theta_1+\theta_2) \quad (2)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 a_1 a_2 (\dot{\theta}_1^2+\dot{\theta}_1\dot{\theta}_2)\sin\theta_2 - m_2 g a_2 \cos(\theta_1+\theta_2) \quad (4)$$

Finally,

$$① - ③ = \tau_1$$

For θ_1,

$$\frac{d\frac{\partial L}{\partial \dot{\theta}_1}}{dt} - \frac{\partial L}{\partial \theta_1} = \tau_1$$

For θ_2,

$$\frac{d\frac{\partial L}{\partial \dot{\theta}_2}}{dt} - \frac{\partial L}{\partial \theta_2} = \tau_2$$

$$② - ④ = \tau_2$$

$$\underbrace{\left[(m_1+m_2)a_1^2+m_2a_2^2+2m_2a_1a_2\cos\theta_2\right]}_{\triangleq A}\ddot{\theta}_1+\underbrace{m_2ga_2\cos(\theta_1+\theta_2)+(m_1+m_2)ga_1\cos\theta_1}_{\triangleq g_1}$$

$$+\underbrace{\left[m_2a_2^2+m_2a_1a_2\cos\theta_2\right]}_{\triangleq B}\ddot{\theta}_2-\underbrace{m_2a_1a_2\left(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^2\right)\sin\theta_2}_{\triangleq v_1}=\tau_1$$

$$\underbrace{\left[m_2a_2^2+m_2a_1a_2\cos\theta_2\right]}_{\triangleq C}\ddot{\theta}_1+\underbrace{m_2a_2^2}_{\triangleq D}\ddot{\theta}_2+\underbrace{m_2a_1a_2\dot{\theta}_1^2}_{\triangleq v_2}$$

$$+\underbrace{m_2ga_2\cos(\theta_1+\theta_2)}_{\triangleq g_2}=\tau_2$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}+\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}+\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}=\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$=$$

These manipulator dynamics are in the standard form

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

inertia matrix

centripeal vector

gravity vector