Control of Robot Manipulators

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CHAPTER 2

Robot Dynamics

This chapter provides the background required for the study of robot manipulator control. The arm dynamical equations are derived both in the second-order differential equation formulation and several state-variable formulations. Some important properties of the dynamics are introduced. We show how to include the dynamics of the arm actuators, which may be electric or hydraulic motors.

2.1 Introduction

Robotics is a complex field involving many diverse disciplines, such as physics, properties of materials, statics and dynamics, electronics, control theory, vision, signal processing, computer programming, and manufacturing. In this book our main interest is control of robot manipulators. The purpose of this chapter is to study the dynamical equations needed for the study of robot control.

For those desiring a background in control theory, Chapter 1 is provided. For those desiring a background in the basics of robot manipulators, in Appendix A we examine the geometric structure of robot manipulators, covering basic manipulator configurations, kinematics, and inverse kinematics. There we review as well the manipulator Jacobian, which is essential for control in Cartesian or workspace coordinates, where the desired trajectories of the arm are usually specified to begin with.

The robot dynamics are derived in Section 2.2. Lagrangian mechanics are used in this derivation. In Section 2.3 we review some fundamental properties of the arm dynamical equation that are essential in subsequent chapters for the derivation of robot control schemes. These are summarized in Table 2.3-1, which is referred to throughout the text.

The arm dynamics in Section 2.2 are in the form of a second-order vector differential equation. In Section 2.4 we show several ways to convert this formulation to a *state-variable description*. The state-variable description is a first-order vector differential equation that is extremely useful for developing many arm control schemes. Feedback linearization techniques and Hamiltonian mechanics are used in this section.

The robot arm dynamics in Section 2.2 are given in joint-space coordinates. In Section 2.5 we show a very general approach to obtaining the arm dynamical description in any desired coordinates, including Cartesian or workspace coordinates and the coordinates of a camera frame or reference.

In Section 2.6 we analyze the electrical or hydraulic actuators that perform the work required to move the links of a robot arm. It is shown how to incorporate dynamical models for the actuators into the arm dynamics to provide a complete dynamical description of the arm-plus-actuator system. This finally leaves us in a position to move on to the next chapters, where robot manipulator control design is discussed.

2.2 Lagrange–Euler Dynamics

For control design purposes, it is necessary to have a mathematical model that reveals the dynamical behavior of a system. Therefore, in this section we derive the dynamical equations of motion for a robot manipulator. Our approach is to derive the kinetic and potential energy of the manipulator and then use Lagrange's equations of motion.

In this section we ignore the dynamics of the electric or hydraulic motors that drive the robot arm; actuator dynamics is covered in Section 2.6.

Force, Inertia, and Energy

Let us review some basic concepts from physics that will enable us to better understand the arm dynamics [Marion 1965]. In this subsection we use boldface to denote vectors and normal type to denote their magnitudes.

The *centripetal force* of a mass m orbiting a point at a radius ω and angular velocity w is given by

$$F_{\text{cent}} = \frac{mv^2}{r} = m\omega^2 r = m\dot{\theta}^2 r. \tag{2.2-1}$$

See Fig. 2.2-1. The linear velocity is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \tag{2.2-2}$$

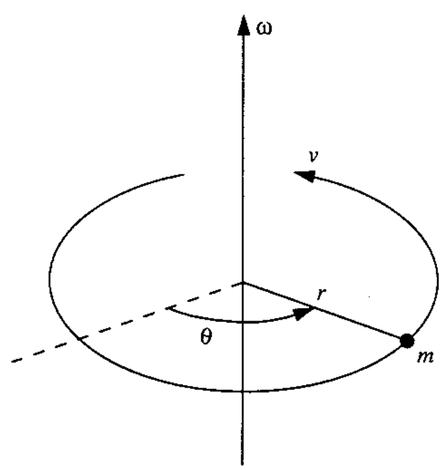

FIGURE 2.2-1 Centripetal force.

which in this case means simply that $v = \omega r$.

Imagine a sphere (i.e., the earth) rotating about its center with an angular velocity of $\boldsymbol{\omega}_0$. See Fig. 2.2-2. The *Coriolis force* on a body of mass m moving with velocity $\mathbf{v}$ on the surface of the sphere is given by

$$\mathbf{F}_{cor} = -2m\boldsymbol{\omega}_0 \times \mathbf{v}. \tag{2.2-3}$$

Using the right-handed screw rule (i.e., if the fingers rotate ω_0 into $\mathbf{v}$, the thumb points in the direction of $\boldsymbol{\omega}_0 \times \mathbf{v}$), we see that, in the figure, the Coriolis force acts to deflect m to the right.

In a low-pressure weather system, the air mass moves toward the center of the low. The Coriolis force is responsible for deflecting the air mass to the right and so causing a counterclockwise circulation known as cyclonic flow. The result is the swirling motion in a hurricane. A brief examination of Fig. 2.2.2 reveals that in the southern hemisphere $\mathbf{F}_{cor}$ deflects a moving mass to the left, so that a low-pressure system would have a clockwise wind motion.

Since $\omega_0 = \dot{\theta}$ and $v = R\dot{\varphi}$, we may write

$$F_{cor} = -2m\dot{\theta}\dot{\varphi}R\,\sin(90° + \varphi) = -2mR\dot{\theta}\dot{\varphi}\,\cos\varphi. \tag{2.2-4}$$

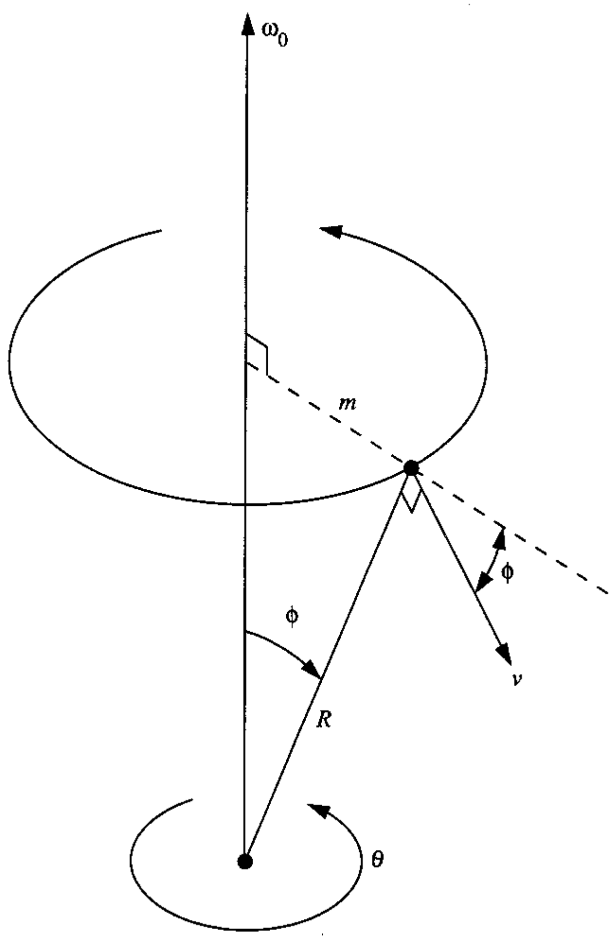

FIGURE 2.2-2 Coriolis force.

It is important to note that the centripetal force involves the square of a single angular velocity, while the Coriolis force involves the product of two distinct angular velocities.

The *kinetic energy* of a mass moving with a linear velocity of v is

$$K = \tfrac{1}{2}mv^2. \tag{2.2-5}$$

The *rotational kinetic energy* of the mass in Fig. 2.2-1 is given by

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2, \tag{2.2-6}$$

where the *moment of inertia* is

$$I = \int_{\text{vol}} \rho\,(r)\, r^2\, dr, \tag{2.2-7}$$

with $\rho(r)$ the mass distribution at radius r in a volume. In the simple case shown where m is a point mass, this becomes

$$I = mr^2. \tag{2.2-8}$$

Therefore,

$$K_{\text{rot}} = \tfrac{1}{2}mr^2\dot{\theta}^2. \tag{2.2-9}$$

The *potential energy* of a mass m at a height h in a gravitational field with constant g is given by

$$P = mgh. \tag{2.2-10}$$

The origin, corresponding to zero potential energy, may be selected arbitrarily since only differences in potential energy are meaningful in terms of physical forces.

The *momentum* of a mass m moving with velocity v is given by

$$\mathbf{p} = m\mathbf{v}. \tag{2.2-11}$$

The *angular momentum* of a mass m with respect to an origin from which the mass has distance r is

$$\mathbf{p}_{\text{ang}} = \mathbf{r} \times \mathbf{p}. \tag{2.2-12}$$

The *torque* or *moment* of a force $\mathbf{F}$ with respect to the same origin is defined to be

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}. \tag{2.2-13}$$

Lagrange's Equations of Motion

Lagrange's equation of motion for a conservative system are given by [Marion 1965]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau, \tag{2.2-14}$$

where q is an n-vector of generalized coordinates q_i, τ is an n-vector of generalized forces τ_i, and the *Lagrangian* is the difference between the kinetic and potential energies

$$L = K - P. \tag{2.2-15}$$

In our usage, q will be the joint-variable vector, consisting of joint angles θ_i (in degrees or radians) and joint offsets d_i (in meters). Then τ is a vector that has components n_i of torque (newton-meters) corresponding to the joint angles, and f_i of force (newtons) corresponding to the joint offsets. Note that we denote the scalar components of τ by lowercase letters.

We shall use Lagrange's equation to derive the general robot arm dynamics. Let us first get a feel for what is going on by considering some examples.

EXAMPLE 2.2-1: Dynamics of a Two-Link Polar Arm

The kinematics for a two-link planar revolute/prismatic (RP) arm are given in Example A.2-3. To determine its dynamics examine Fig. 2.2-3, where the joint-variable and joint-velocity vectors are

$$q = \begin{bmatrix} \theta \\ r \end{bmatrix}, \qquad \dot{q} = \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix}. \tag{1}$$

The corresponding generalized force vector is

$$\tau = \begin{bmatrix} n \\ f \end{bmatrix} \tag{2}$$

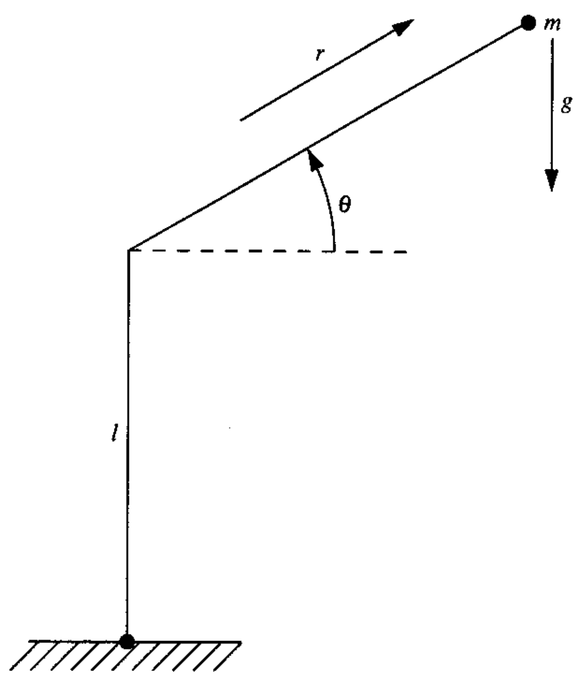

FIGURE 2.2-3 Two-link planar RP arm.

with n a torque and f a force. The torque n and force f may be provided by either motors or hydraulic actuators. We discuss the dynamics of actuators in Section 2.6.

To determine the arm dynamics, we must now compute the quantities required for the Lagrange equation.

a. Kinetic and Potential Energy

The total kinetic energy due to the angular motion $\dot{\theta}$ and the linear motion $\dot{r}$ is

$$K = \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{r}^2 \tag{3}$$

and the potential energy is

$$P = mgr\sin\theta. \tag{4}$$

b. Lagrange's Equation

The Lagrangian is

$$L = K - P = \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{r}^2 - mgr\sin\theta. \tag{5}$$

Now we obtain

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \dfrac{\partial L}{\partial \dot{\theta}} \\ \dfrac{\partial L}{\partial \dot{r}} \end{bmatrix} = \begin{bmatrix} mr^2\dot{\theta} \\ m\dot{r} \end{bmatrix} \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} \\ m\ddot{r} \end{bmatrix} \tag{7}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} -mgr\cos\theta \\ mr\dot{\theta}^2 - mg\sin\theta \end{bmatrix}. \tag{8}$$

Therefore, (2.2-14) shows that the arm dynamical equations are

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta = n \tag{9}$$

$$m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = f. \tag{10}$$

This is a set of *coupled nonlinear differential equations* which describe the motion $q(t) = [\theta(t) \quad r(t)]^T$ given the control input torque $n(t)$ and force $f(t)$. We shall show how to determine $q(t)$ given the control inputs $n(t)$ and $f(t)$ by *computer simulation* in Chapter 3.

Given our discussion on forces and inertias it is easy to identify the terms in the dynamical equations. The first terms in each equation are acceleration terms involving *masses and inertias.* The second term in (9) is a *Coriolis term,* while the second term in (10) is a *centripetal term.* The third terms are *gravity* terms.

c. Manipulator Dynamics

By using vectors, the arm equations may be written in a convenient form. Indeed, note that

$$\begin{bmatrix} mr^2 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} 2mr\dot{r}\dot{\theta} \\ -mr\dot{\theta}^2 \end{bmatrix} + \begin{bmatrix} mgr\cos\theta \\ mg\sin\theta \end{bmatrix} = \begin{bmatrix} n \\ f \end{bmatrix}. \tag{11}$$

We symbolize this vector equation as

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau. \tag{12}$$

Note that, indeed, the *inertia matrix* $M(q)$ is a function of q (i.e., of θ and r), the *Coriolis/centripetal vector* $V(q,\dot{q})$ is a function of q and $\dot{q}$, and the *gravity vector* $G(q)$ is a function of q.

EXAMPLE 2.2-2: Dynamics of a Two-Link Planar Elbow Arm

In Example A.2-2 are given the kinematics for a two-link planar RR arm. To determine its dynamics, examine Fig. 2.2-4, where we have assumed that the link masses are concentrated at the ends of the links. The joint variable is

$$q = [\theta_1 \quad \theta_2]^T \tag{1}$$

and the generalized force vector is

$$\tau = [\tau_1 \quad \tau_2]^T \tag{2}$$

with τ_1 and τ_2 torques supplied by the actuators.

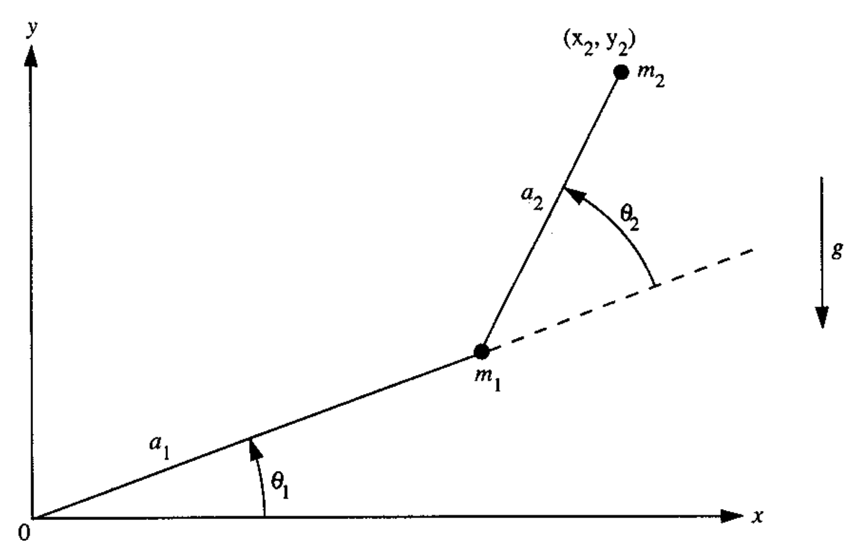

FIGURE 2.2-4 Two-link planar RR arm.

a. Kinetic and Potential Energy

For link 1 the kinetic and potential energies are

$$K_1 = \tfrac{1}{2} m_1 a_1^2 \dot\theta_1^2 \tag{3}$$

$$P_1 = m_1 g a_1 \sin\theta_1 . \tag{4}$$

For link 2 we have

$$x_2 = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \tag{5}$$

$$y_2 = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \tag{6}$$

$$\dot x_2 = -a_1 \dot\theta_1 \sin\theta_1 - a_2(\dot\theta_1 + \dot\theta_2) \sin(\theta_1 + \theta_2) \tag{7}$$

$$\dot y_2 = a_1 \dot\theta_1 \cos\theta_1 + a_2(\dot\theta_1 + \dot\theta_2) \cos(\theta_1 + \theta_2), \tag{8}$$

so that the velocity squared is

$$v_2^2 = \dot x_2^2 + \dot y_2^2 = a_1^2 \dot\theta_1^2 + a_2^2(\dot\theta_1 + \dot\theta_2)^2 + 2a_1a_2(\dot\theta_1^2 + \dot\theta_1\dot\theta_2)\cos\theta_2 . \tag{9}$$

Therefore, the kinetic energy for link 2 is

$$K_2 = \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_2 a_1^2 \dot\theta_1^2 + \tfrac{1}{2} m_2 a_2^2 (\dot\theta_1 + \dot\theta_2)^2 + m_2 a_1 a_2 (\dot\theta_1^2 + \dot\theta_1\dot\theta_2)\cos\theta_2 . \tag{10}$$

The potential energy for link 2 is

$$P_2 = m_2 g y_2 = m_2 g[a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)]. \tag{11}$$

b. Lagrange's Equation

The Lagrangian for the entire arm is

$$\begin{aligned} L &= K - P = K_1 + K_2 - P_1 - P_2 \\ &= \tfrac{1}{2}(m_1 + m_2)\, a_1^2 \dot\theta_1^2 + \tfrac{1}{2} m_2 a_2^2 (\dot\theta_1 + \dot\theta_2)^2 + m_2 a_1 a_2 (\dot\theta_1^2 + \dot\theta_1\dot\theta_2)\cos\theta_2 \\ &\quad - (m_1 + m_2)\, g a_1 \sin\theta_1 - m_2 g a_2 \sin(\theta_1 + \theta_2). \end{aligned} \tag{12}$$

The terms needed for (2.2-14) are

$$\frac{\partial L}{\partial \dot\theta_1} = (m_1 + m_2)\, a_1^2 \dot\theta_1 + m_2 a_2^2 (\dot\theta_1 + \dot\theta_2) + m_2 a_1 a_2 (2\dot\theta_1 + \dot\theta_2)\cos\theta_2$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} &= (m_1 + m_2)\, a_1^2 \ddot\theta_1 + m_2 a_2^2 (\ddot\theta_1 + \ddot\theta_2) + m_2 a_1 a_2 (2\ddot\theta_1 + \theta_2)\cos\theta_2 \\ &\quad - m_2 a_1 a_2 (2\dot\theta_1\dot\theta_2 + \dot\theta_2^2)\sin\theta_2 \end{aligned}$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2) g a_1 \cos\theta_1 - m_2 g a_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 a_2^2 (\dot\theta_1 + \dot\theta_2) + m_2 a_1 a_2 \dot\theta_1 \cos\theta_2$$

$$\frac{\mathrm{d}}{dt}\frac{\partial L}{\partial \dot\theta_2} = m_2 a_2^2 (\ddot\theta_1 + \ddot\theta_2) + m_2 a_1 a_2 \ddot\theta_1 \cos\theta_2 - m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 a_1 a_2 (\dot\theta_1^2 + \dot\theta_1\dot\theta_2)\sin\theta_2 - m_2 g a_2 \cos(\theta_1 + \theta_2).$$

Finally, according to Lagrange's equation, the arm dynamics are given by the two coupled nonlinear differential equations

$$\begin{aligned}\tau_1 &= [(m_1 + m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2 \cos \theta_2]\, \ddot{\theta}_1 \\ &\quad + [m_2a_2^2 + m_2a_1a_2 \cos \theta_2]\, \ddot{\theta}_2 - m_2a_1a_2\, (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \sin \theta_2 \\ &\quad + (m_1 + m_2)\, ga_1 \cos \theta_1 + m_2ga_2 \cos (\theta_1 + \theta_2)\end{aligned} \tag{13}$$

$$\begin{aligned}\tau_2 &= [m_2a_2^2 + m_2a_1a_2 \cos \theta_2]\, \ddot{\theta}_1 + m_2a_2^2\ddot{\theta}_2 + m_2a_1a_2\dot{\theta}_1^2 \sin \theta_2 \\ &\quad + m_2ga_2 \cos (\theta_1 + \theta_2).\end{aligned} \tag{14}$$

c. Manipulator Dynamics

Writing the arm dynamics in vector form yields

$$\begin{aligned}&\begin{bmatrix} (m_1 + m_2)\, a_1^2 + m_2a_2^2 + 2m_2a_1a_2 \cos \theta_2 & m_2a_2^2 + m_2a_1a_2 \cos \theta_2 \\ m_2a_2^2 + m_2a_1a_2 \cos \theta_2 & m_2a_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \\ &+ \begin{bmatrix} -m_2a_1a_2\,(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \sin \theta_2 \\ m_2a_1a_2\dot{\theta}_1^2 \sin \theta_2 \end{bmatrix} + \begin{bmatrix} (m_1 + m_2)\, ga_1 \cos \theta_1 + m_2ga_2 \cos (\theta_1 + \theta_2) \\ m_2ga_2 \cos (\theta_1 + \theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.\end{aligned} \tag{15}$$

These manipulator dynamics are in the standard form

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau, \tag{16}$$

with $M(q)$ the inertia matrix, $V(q,\dot{q})$ the Coriolis/centripetal vector, and $G(q)$ the gravity vector. Note that $M(q)$ is symmetric.

EXERCISE 2.2-3: Dynamics of a Three-Link Cylindrical Arm

We study the kinematics of a three-link cylindrical arm in Example A.2-1. In Fig. 2.2-5 the joint variable vector is

$$q = [\theta \quad h \quad r]^T. \tag{1}$$

Show that the manipulator dynamics are given by

$$\begin{bmatrix} J + m_2r^2 & & \\ & m_1 + m_2 & \\ & & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{h} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} 2m_2r\dot{r}\dot{\theta} \\ 0 \\ -m_r\dot{\theta}^2 \end{bmatrix} + \begin{bmatrix} 0 \\ (m_1 + m_2)\, gh \\ 0 \end{bmatrix} = \begin{bmatrix} n_1 \\ f_2 \\ f_3 \end{bmatrix}, \tag{2}$$

with J the inertia of the base link and the force vector

$$\tau = [n_1 \quad f_2 \quad f_3]^T. \tag{3}$$

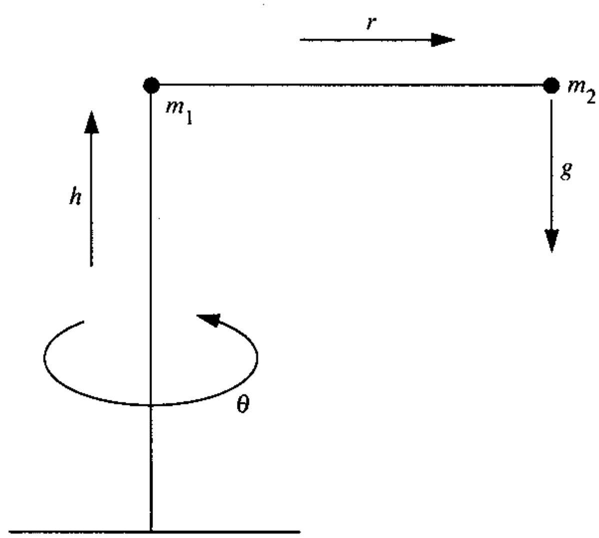

FIGURE 2.2-5 Three-link cylindrical arm.

Derivation of Manipulator Dynamics

We have shown in several examples how to apply Lagrange's equation to compute the dynamical equations of any given robot manipulator. In the examples the dynamics we found always had the special form

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau, \tag{2.2-16}$$

with q the joint-variable vector and τ the generalized force/torque vector. In this subsection we derive the dynamics for a general robot manipulator. They will be of this same form.

To obtain the general robot arm dynamical equation, we determine the arm kinetic and potential energies, then the Lagrangian, and then substitute into Lagrange's equation (2.2-14) to obtain the final result [Paul 1981; Lee et al. 1983; Asada and Slotine 1986; Spong and Vidyasagar 1989].