ELEC 5530/6530 Robotic Control

Lecture 3: Basic Manipulator Geometries And Robot Kinematics

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- **Basic Manipulator Geometries**

- **Robot Kinematics**

Basic Manipulator Geometries

A robot arm composed of a set of ***joints*** separated in space by the arm ***links***.

Joints

- where the motion in the arm occurs

--wrist and elbow

---actuated by motors or hydraulic actuators

Links

- fixed construction

--our own forearm

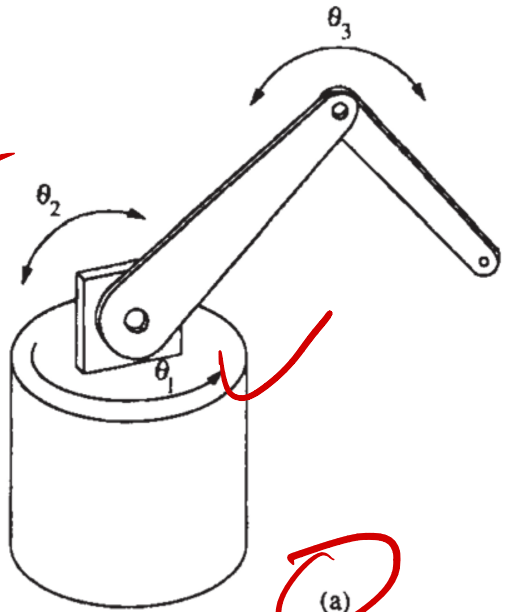
---maintain a fixed relationship between the joints.

1. A *revolute joint* (denoted by *R*) is one that allows rotary motion about an axis of rotation

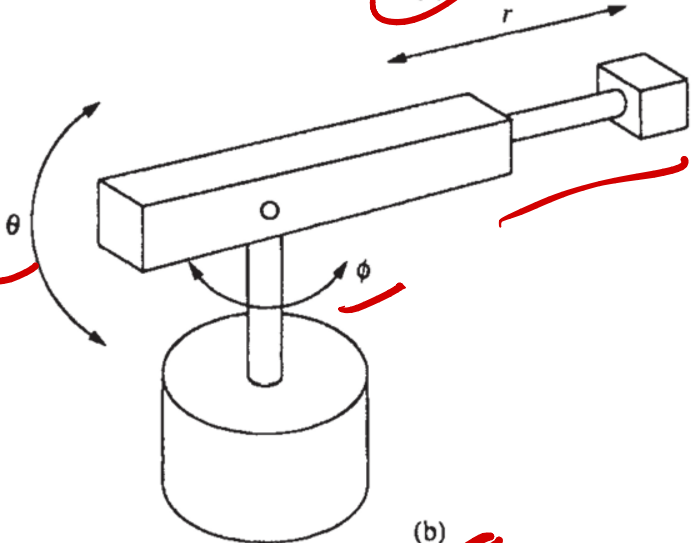
--human elbow

2. A *prismatic joint* (denoted by *P*) is one that allows extension or telescopic motion.

--telescoping automobile antenna.



(a)



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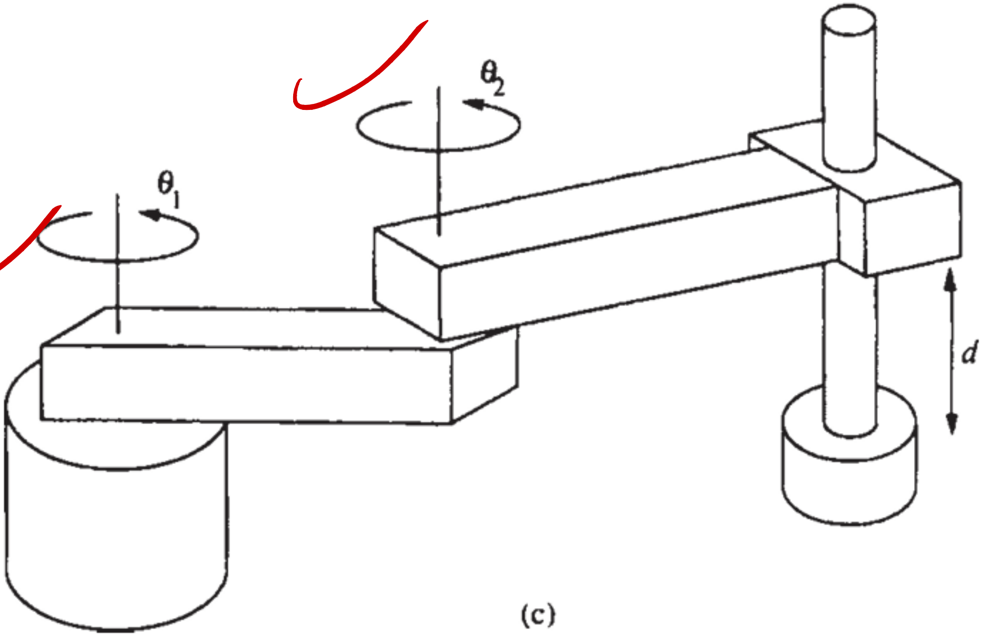

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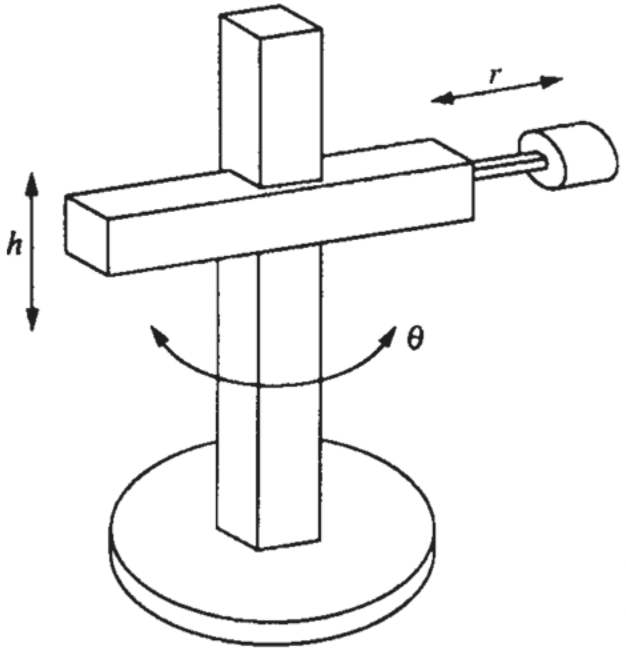

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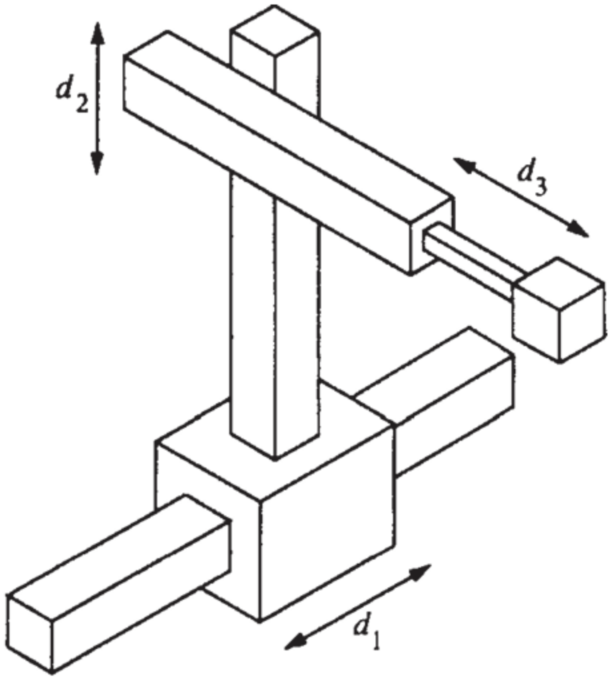


(e)

- Many industrial robots are ***serial link*** manipulators since they consist of ***a series of links*** connected together by actuated joints.
- The base is called link ***0***, and the last link is terminated by the ***tool*** or ***end effector***.
- Many robots have six joints, corresponding to the ***six degrees of freedom*** needed to obtain ***arbitrary position*** and ***orientation*** of the end effector in three-dimensional space.

DOF

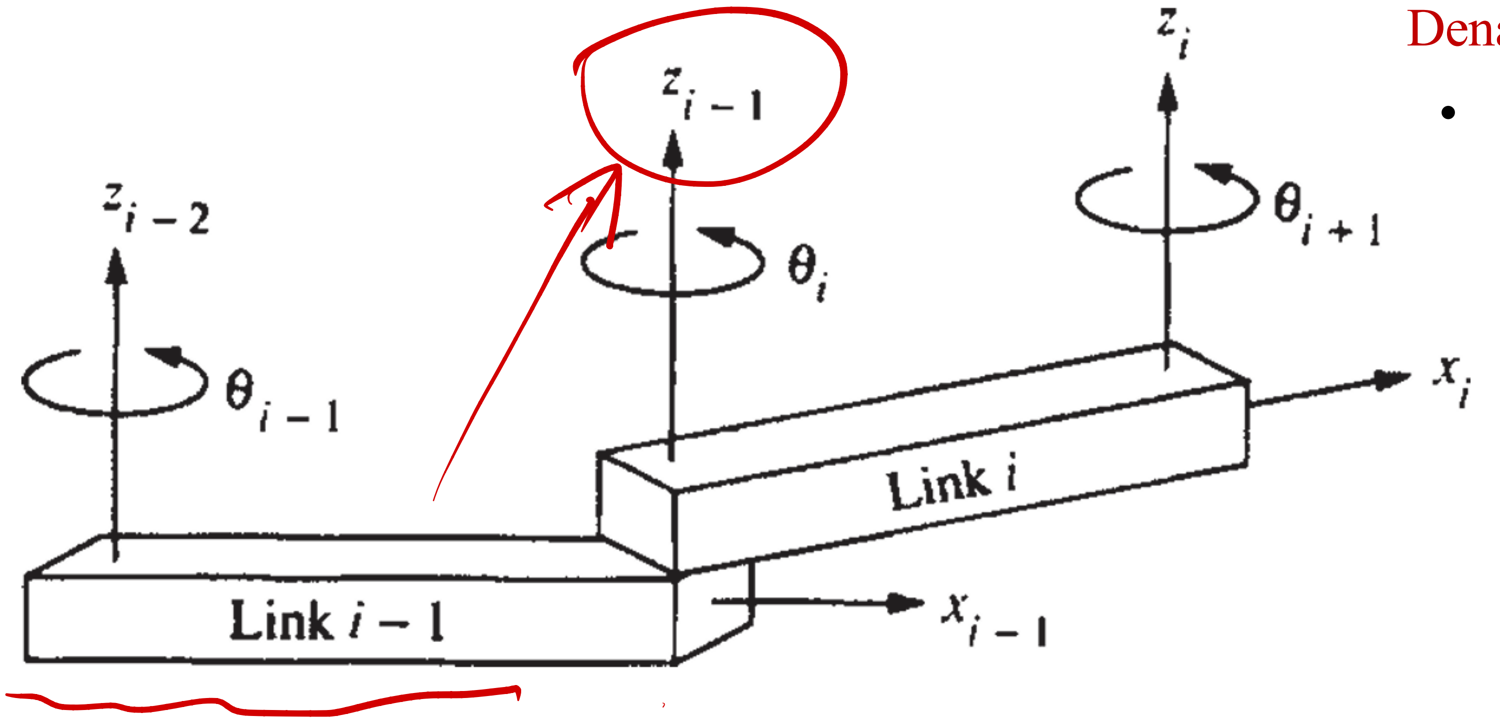
Robot Kinematics

- Arm A matrices
- Homogeneous transformations
- T matrix
- Forward and inverse kinematics
- Joint-space
- Cartesian coordinates

Arm A matrices

- For given values of the joint variables, it is important to be able to specify the locations of the links with respect to each other. This is accomplished by using the manipulator kinematic equations.

- We associate with each link i a coordinate frame (x_i , y_i , z_i) fixed to that link.



Denavit-Hartenberg (*D-H*) representation

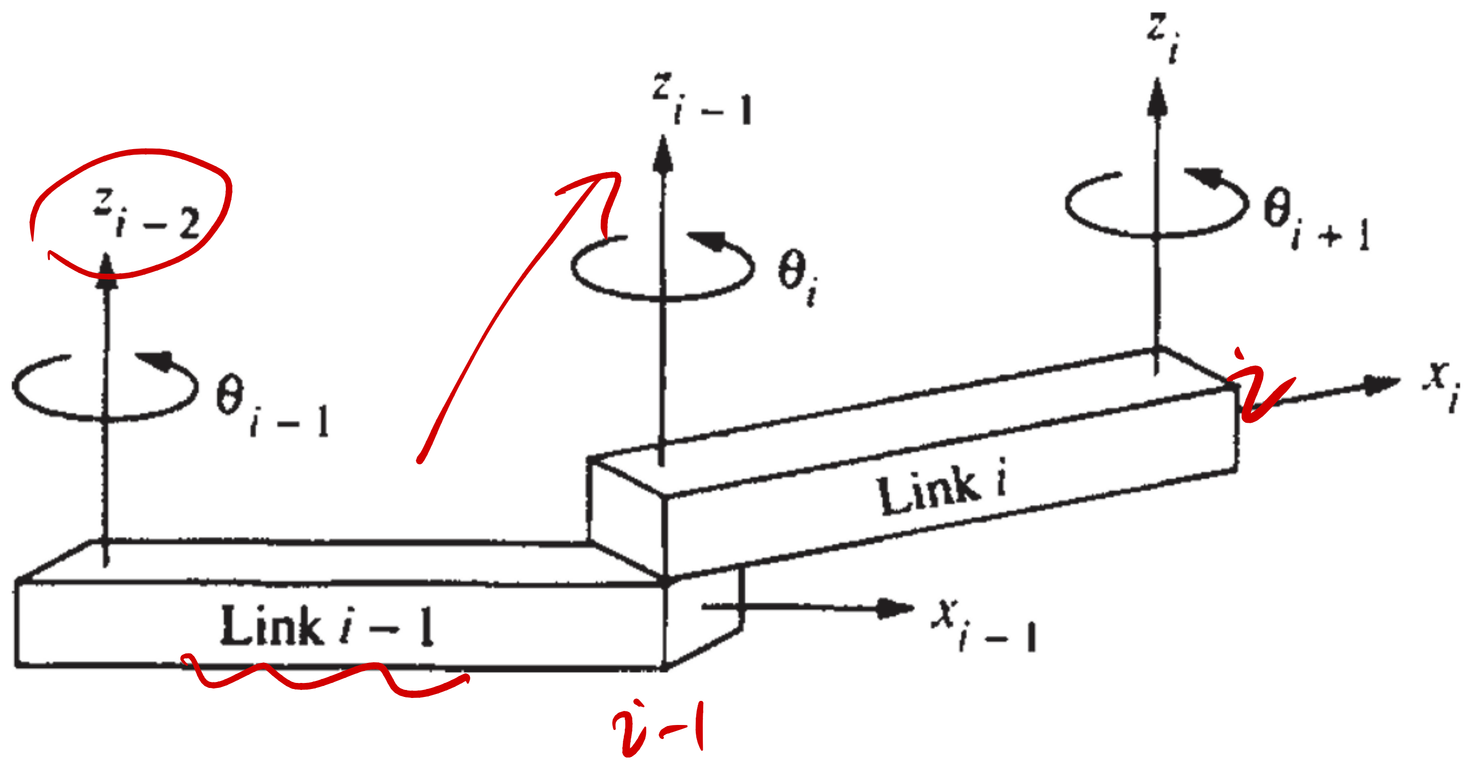
- The frame attached to link 0 (i.e., the base of the manipulator) is called the base frame or inertial frame.

The relation between coordinate frame i-1 and coordinate frame i is given by the transformation matrix

$$A_i = \left[\begin{array}{ccc|c} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

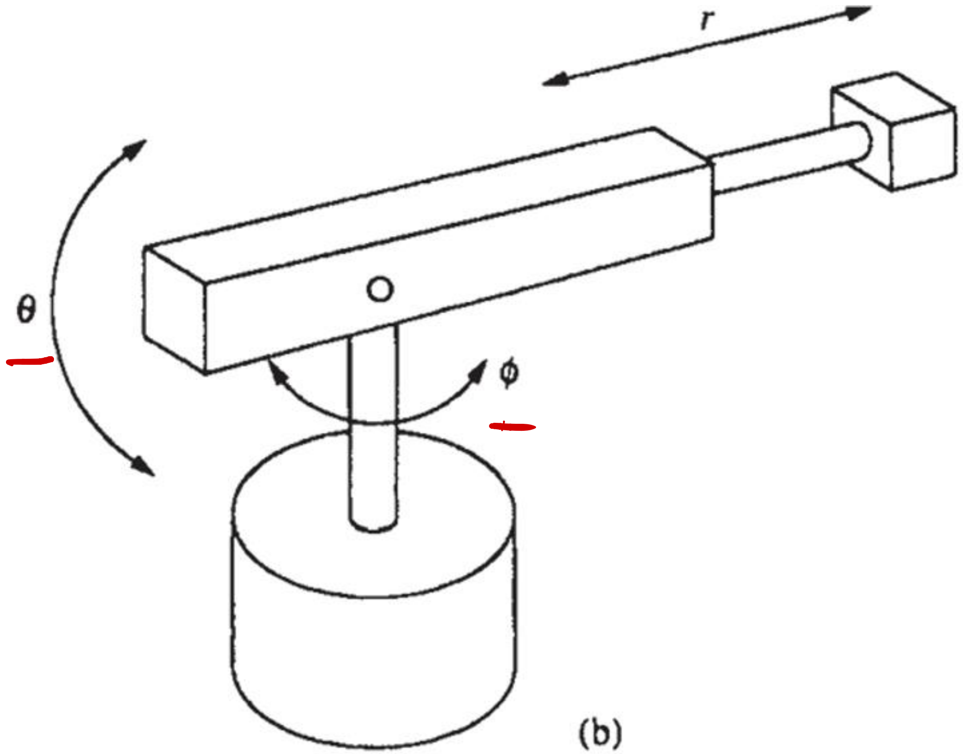
$$A_i = \begin{bmatrix} \underline{R_i} & \underline{P_i} \\ 0 & 1 \end{bmatrix}$$

By the *D-H* convention, for a revolute joint, the rotation i occurs about axis z_{i-1}. For a prismatic joint, d_i occurs along axis z_{i-1}. Thus the link coordinate frame is considered to be attached to the ***outer end*** of the link.



- The A matrix A_i is a function of only a single variable, namely the joint variable θ_i or d_i and all the other parameters in A_i are fixed for a specific joint.
- If a manipulator has n links, the joint-variable vector q is an n-vector composed of the individual joint variables.
- q is in general a combination of angles i and lengths d_i. For instance, for an RRP arm.

$$q = [\theta_1 \quad \theta_2 \quad d_3]^T$$



(b)

Homogeneous Transformations

The A matrix is a homogeneous transformation matrix of the form

$$A_i = \begin{bmatrix} R_i & P_i \\ 0 & 1 \end{bmatrix}, \qquad ①$$

where R_i is a rotation matrix and p_i is a translation vector. Thus, if r_i is a point described with respect to the coordinate frame of link i, the same point has coordinates r_{i-1} with respect to the frame of link i-1 given by

$$r_{i-1} = A_i r_i \qquad ②$$

Here $r_{i-1} = \begin{bmatrix} x_{i-1} \\ y_{i-1} \\ z_{i-1} \\ 1 \end{bmatrix}$ $r_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$

Based on ②, we have

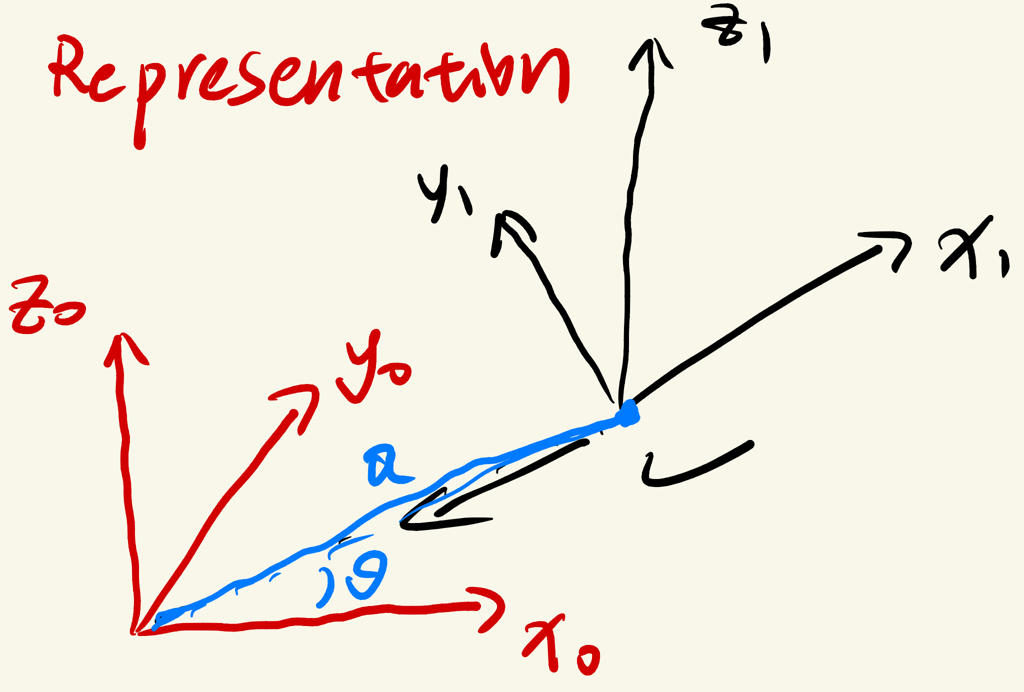
$$\begin{bmatrix} x_{i-1} \\ y_{i-1} \\ z_{i-1} \\ 1 \end{bmatrix} = \begin{bmatrix} \underbrace{R_i}_{3\times 3} & \underbrace{P_i}_{3\times 1} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

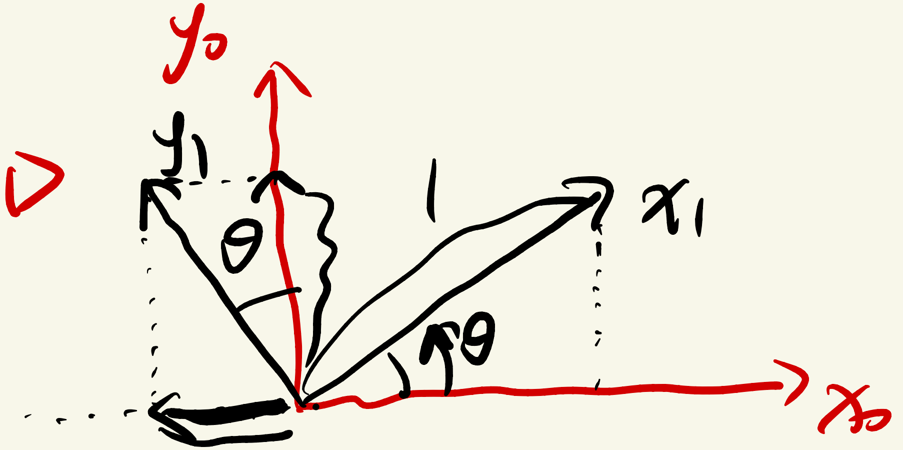
$$\begin{bmatrix} x_{i-1} \\ y_{i-1} \\ z_{i-1} \end{bmatrix} = R_i \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + P_i$$

How to find R_i ?

- Interpret frame *i*-1 as the fixed (i.e., “original”) frame and frame *i* as the rotated and translated (i.e., “new”) frame.
- According to (2), there is an easy way to find the rotation matrix R_i that rotates a given coordinate frame *i*-1 into another given frame *i*.
- Set p_i equal to zero and $(x_i\ , y_i\ , z_i\ )$=(1,0,0). Then (2) is equal to the first column of R_i. That is, ***the first column of*** R_i ***is the representation of the new x-axis*** x_i ***in terms of the original coordinates*** $(x_{i-1}\ , y_{i-1}, z_{i-1}\ )$. Similarly, the second (respectively third) column of R_i is the representation of the rotated *y*-axis y_i (respectively, *z*-axis and z_i) in the fixed frame *i*-1.

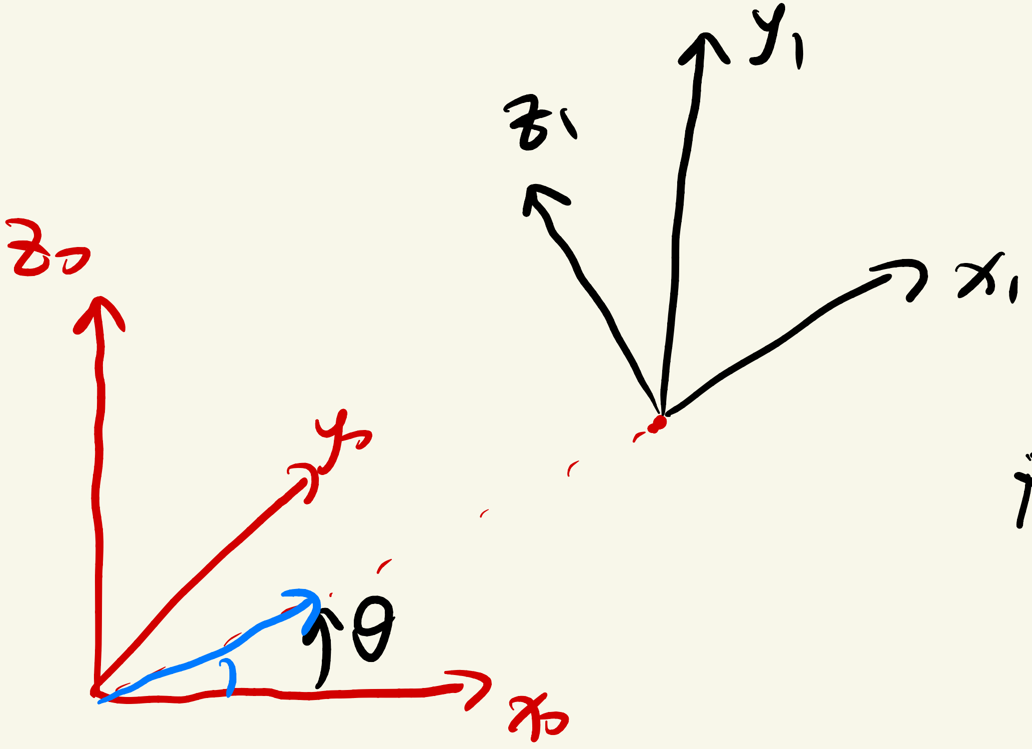
Representation





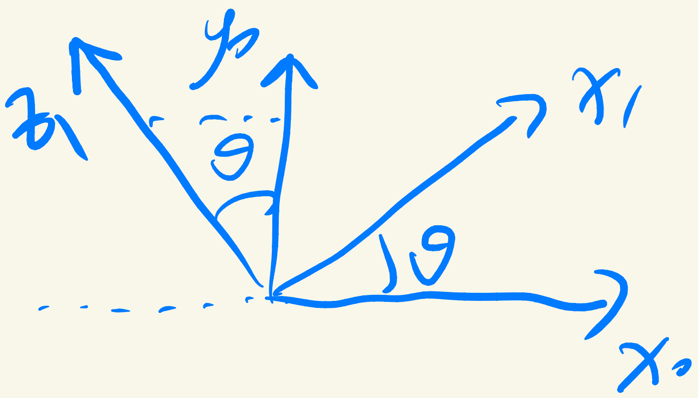
$$R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} a\cos\theta \\ a\sin\theta \\ 0 \end{bmatrix}$$

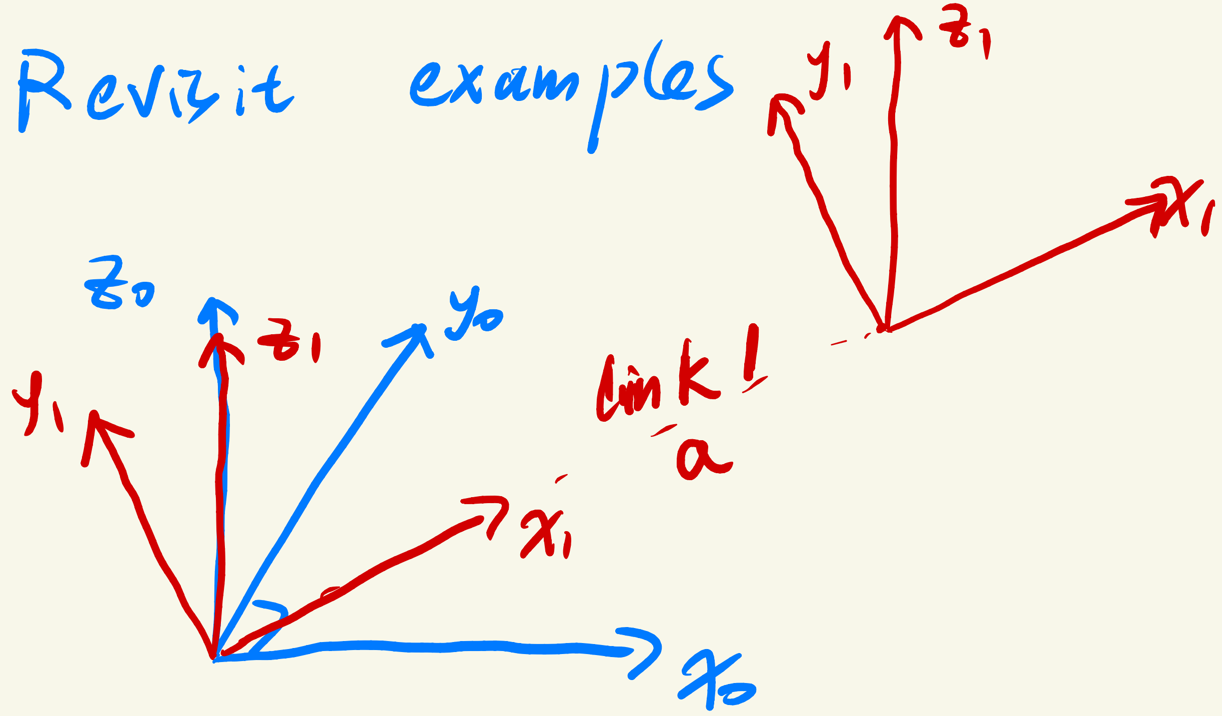


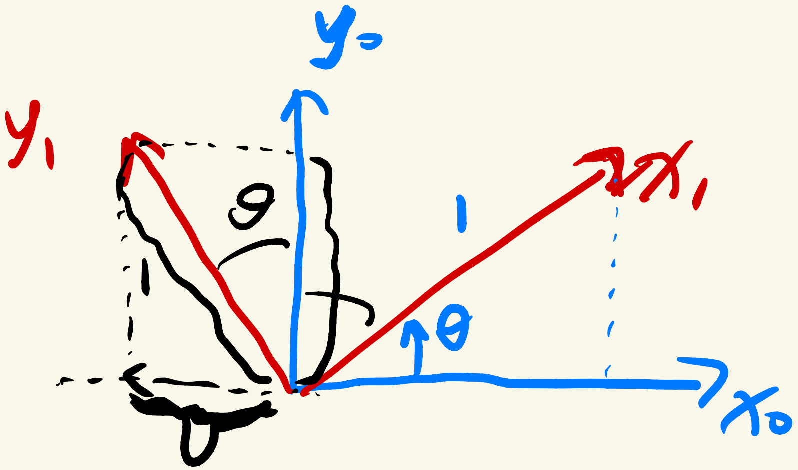
$$R_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_1 = [a\cos\theta \quad a\sin\theta \quad 0]^T \checkmark$$



Revisit examples





$$
R_1 = \begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \\ \end{array}
$$

$$
R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

the representation of x_1 on x_0 is $\cos\theta$

the representation of x_1 on y_0 is $\sin\theta$

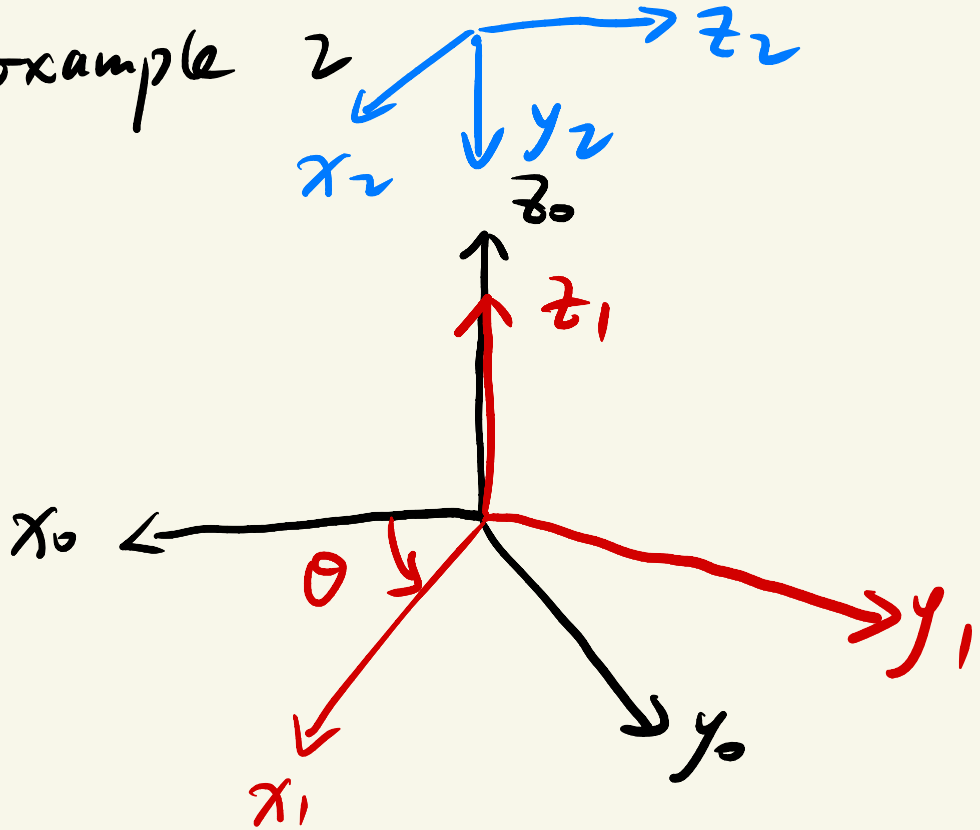
the representation of x_1 on z_0 is 0

. . . . of y_1 on x_0 is $-\sin\theta$

. . . . of y_1 on y_0 is $\cos\theta$

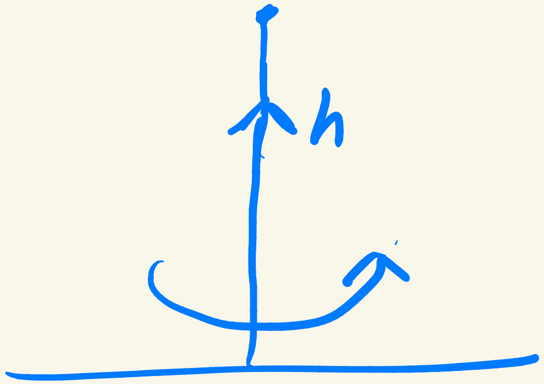
on z_0 is 0

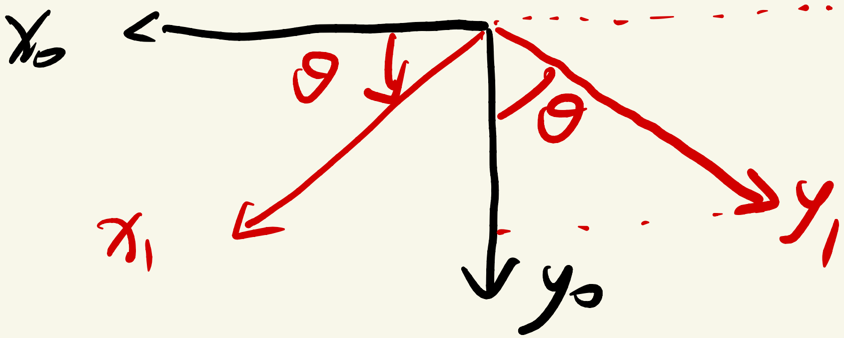
Example 2



$$R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

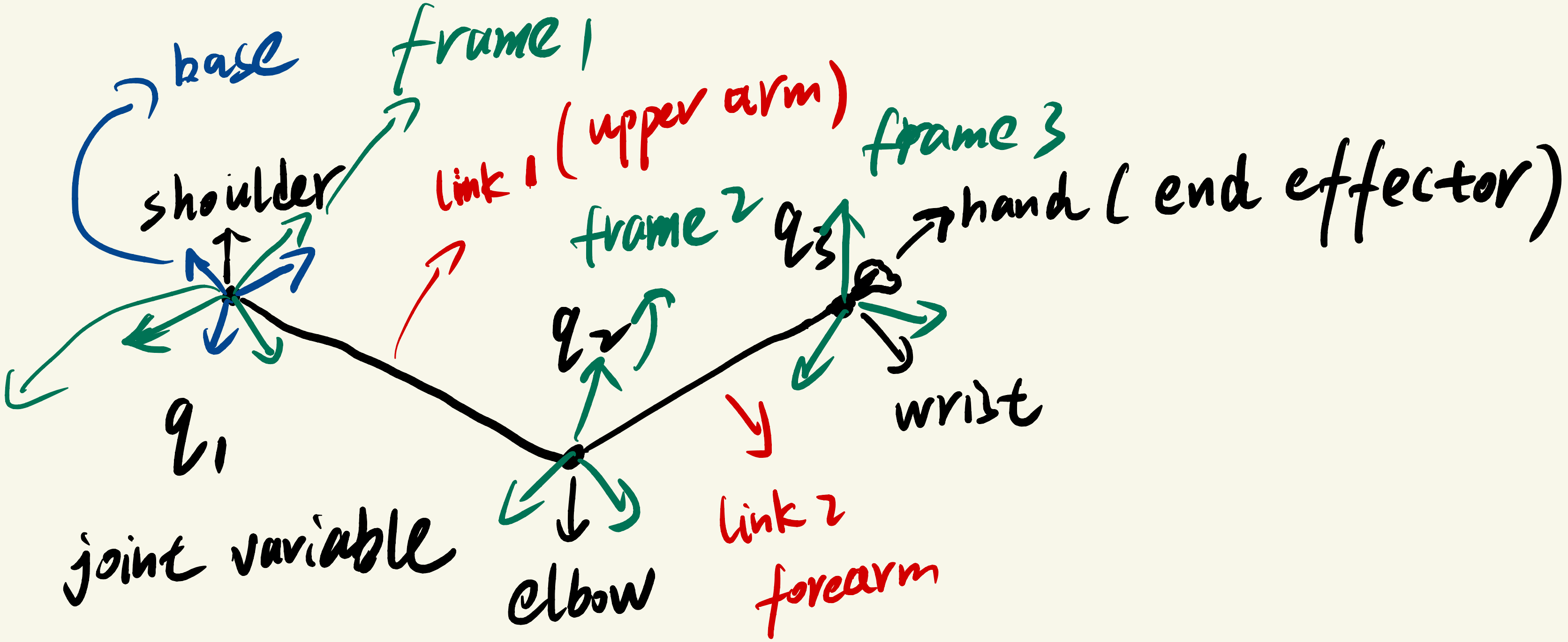
$$R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$





A_i may be interpreted from several points of view.

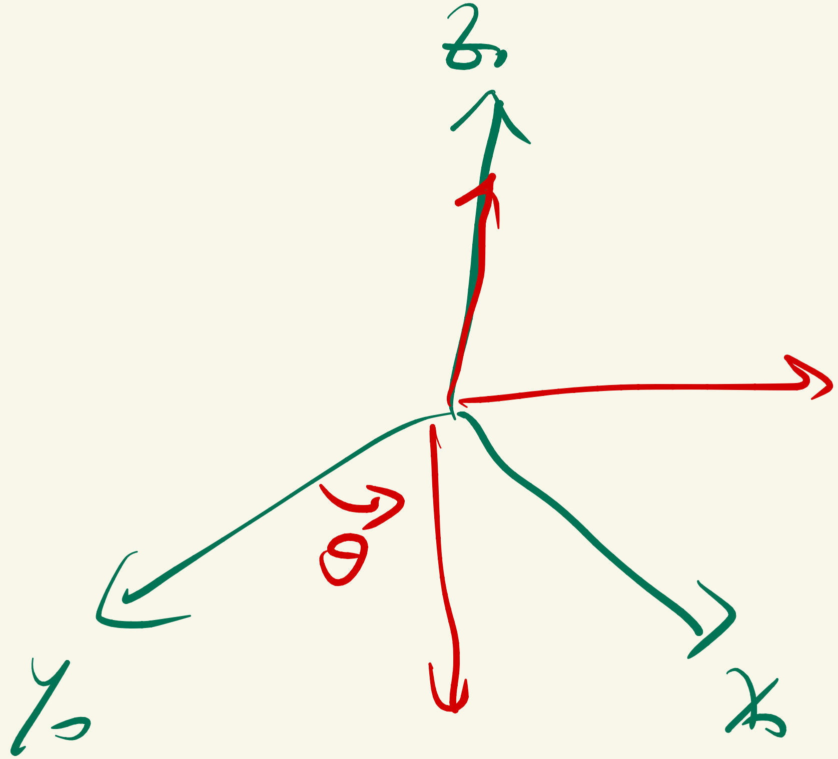
- It is the transformation that takes a representation r_i of a vector in frame i to its representation r_{i-1} in frame *i*-1.
- It is the description of frame i in terms of frame *i*-1;
- In fact, R_i describes the orientation of the axes of frame i in terms of frame *i*-1, while p_i describes the origin of frame i in terms of the coordinates of frame *i*-1.



$$r_0 = A_1 r_1$$

$$r_1 = A_2 r_2$$

$$r_2 = A_3 r_3$$



$$
\begin{aligned}
r_0 &= A_1 A_2 r_2 \\
&= \underbrace{A_1 A_2 A_3}_{\text{kinematic chain}} r_3
\end{aligned}
$$

T matrix

To obtain the coordinates of a point in terms of the base (i.e., link 0) frame, we may use the matrices

$$T_i = A_1 \times A_2 \times \cdots \times A_i$$

Then, given the coordinates r_i of a point expressed in the frame attached to link i, the coordinates of the same point in the base frame are given by

$$r_0 = T_i r_i$$

We define the *arm T matrix* as

$$T \equiv T_n = A_1 \times A_2 \times \cdots \times A_n$$

with n the number of links in the manipulator. Then, if r_n are the coordinates of a point referred to the last link, the base coordinates of the point are

$$r_0 = T r_n$$

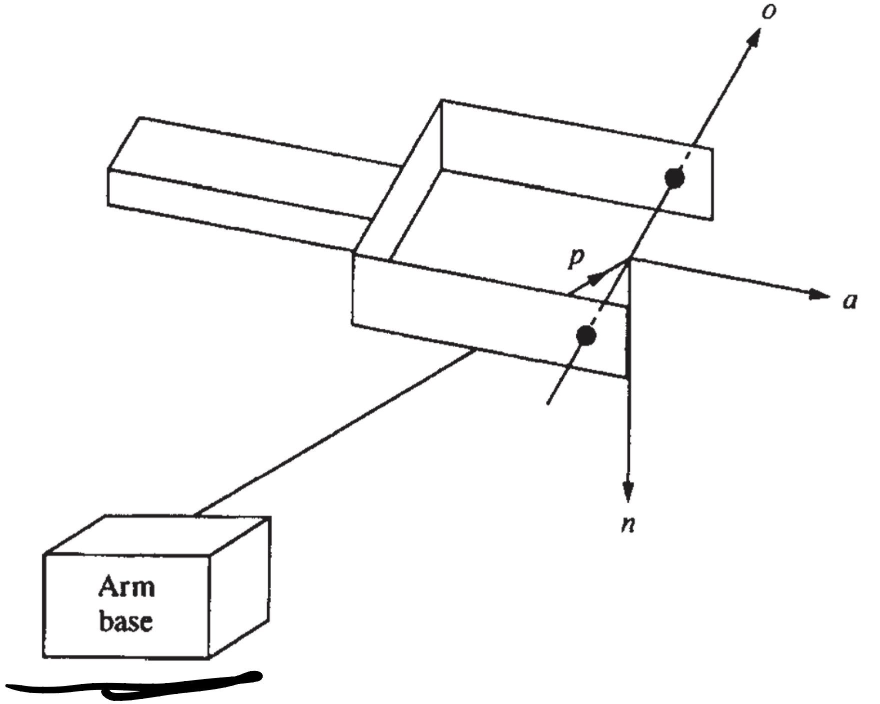
r_0 represents the location of the object with respect to the base frame, which is the object's absolute position with respect to the manipulator base.

Forward Kinematics

The position and orientation of the end effector with respect to the manipulator base frame are given by evaluating the arm T matrix. It is conventional to symbolize this homogeneous transformation as

$$T = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the orientations of the axes of the end-effector reference frame are described with respect to base coordinates by the rotation matrix $R = [n\ o\ a]$, and the origin of the end-effector frame has a position of p in base coordinates.



The 3-vectors n o a and p are defined as in the right figure. The *approach* vector of the end effector is a; the *orientation* vector, o, is the direction specifying the orientation of the hand, from fingertip to fingertip. The *normal* vector n is chosen to complete the definition of **a right-handed coordinate system using**

$$n = 0 \times a$$

thumb → x → n

index finger → y → o

middle finger → z → a

Thus (n, o, a) are the base coordinates of an (x, y, z) *Cartesian coordinate system* attached to the end effector. The position vector p specifies the location of the origin of the (n, o, a) frame with respect to the base frame.

- The representation (n, o, a) for the orientation of the end effector is inefficient.
- Note that $[n\ o\ a]$ is a 3×3 matrix, so that it has nine entries.
- Alternative more efficient methods of specifying the orientation of the end effector in base coordinates are the roll-pitch-yaw, Euler angle, quaternion, and tool-configuration vector descriptions. We discuss some of these later.
- The reason we use (n, o, a) is that the arm T matrix is easily computed using the A matrices in terms of the arm parameters and join variables. Then n, o, a, and p are simply read off by examining the T matrix, as we shall see in the examples.

Joint variable coordinates

At this point, we should distinguish between two sets of coordinates describing the end effector. The ***joint variable coordinates*** of the end-effect are given by the n-vector

$$q = [q_1 \quad q_2 \quad \cdots \quad q_n]^T$$

where q_i can represent angles or lengths, depending on whether the links are revolute or prismatic.

q is the joint-space description of the position and orientation of the end effector, while *(n, o, a, p) is the Cartesian-space* description. This terminology derives from the fact that descriptions of tasks to be performed by an arm are generally given in Cartesian coordinates, not in joint coordinates.

kinematics problem

P R

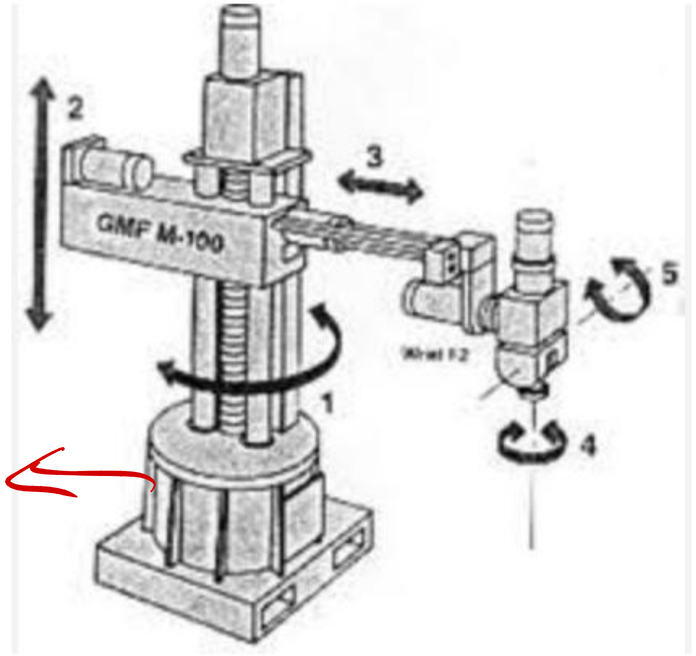
Given the joint variables q, find the Cartesian position and orientation of the end of the manipulator. Thus the kinematics problem amounts to converting given joint variables into the Cartesian position and orientation of the en⁶d effector expressed in base coordinates. It amounts to computing T.

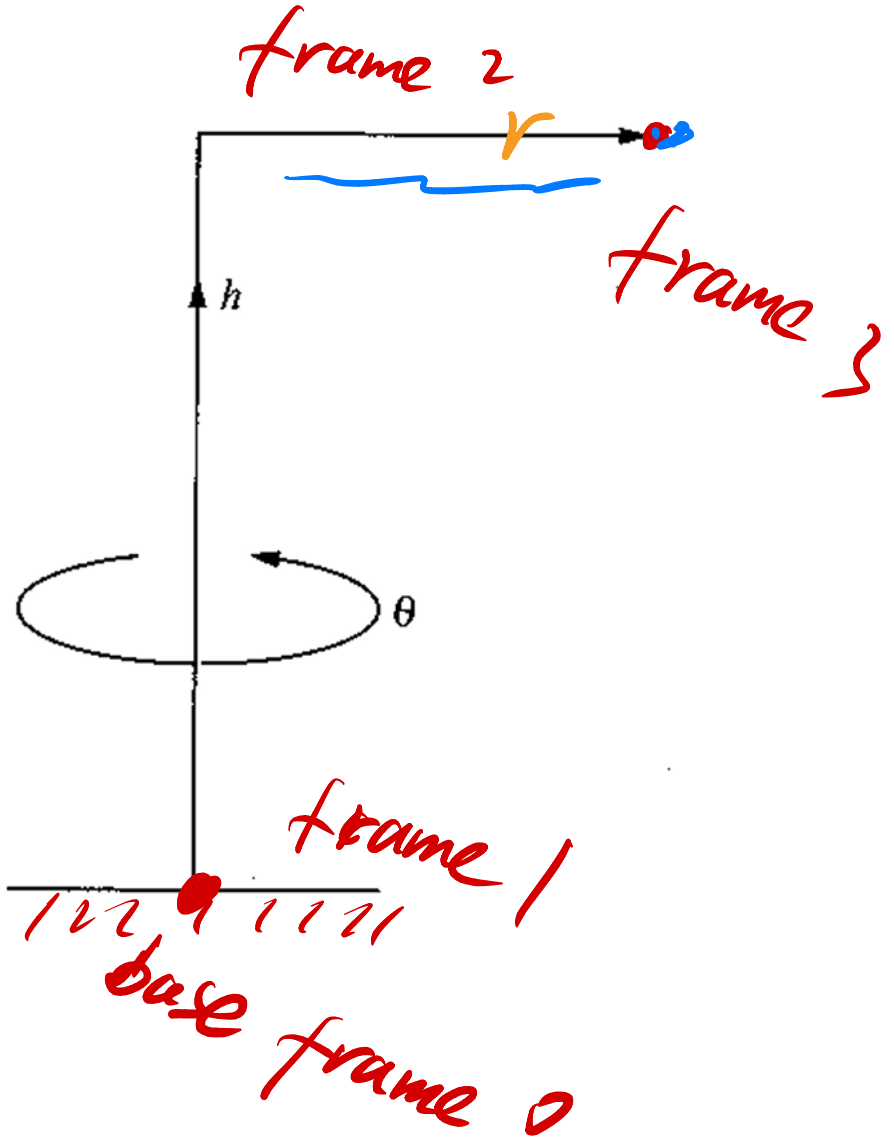
$$T = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$$

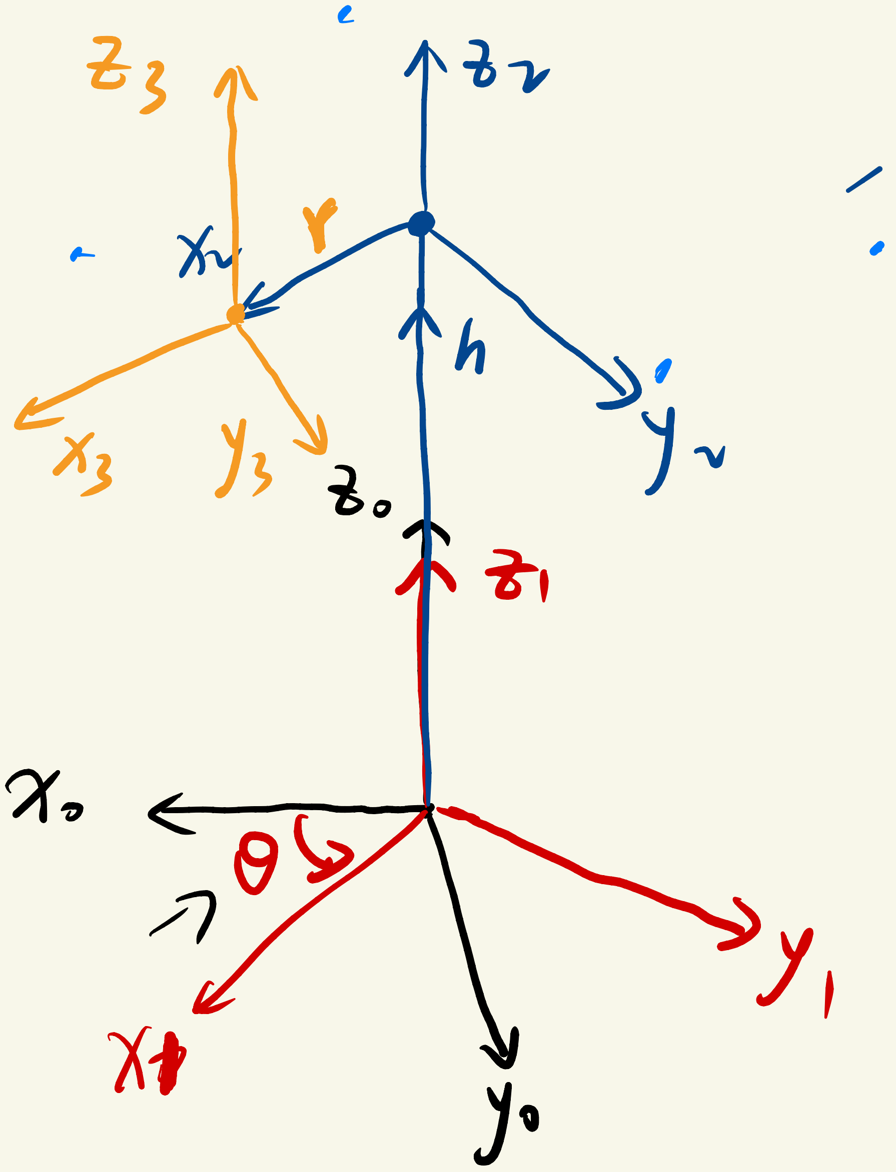
Example 1: Kinematics for Three-Link Cylindrical Arm

Given joints

find T

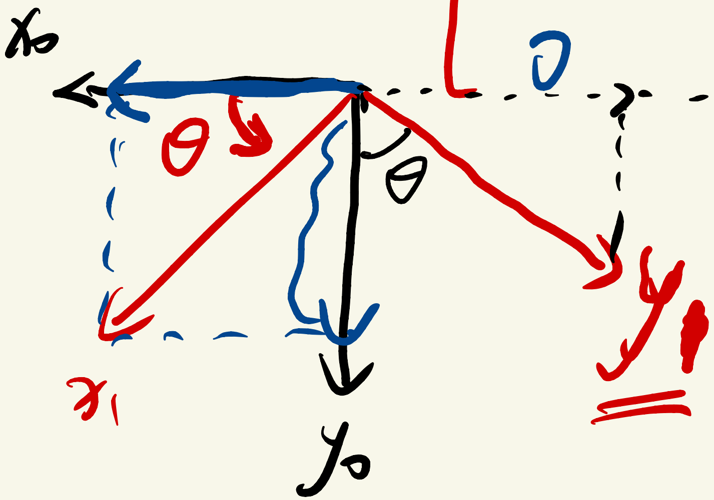






$$A_1 = \begin{bmatrix} R_1 & P_1 \\ 0 & 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$P_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} n_1 & o_1 & a_1 & p_1 \end{array} \\
A_1 = & \left[\begin{array}{ccc:c}
C\theta & -S\theta & 0 & 0 \\
S\theta & C\theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hdashline
0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$$
R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}
$$

$$A_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$A_3 = \left[\begin{array}{ccc:c} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$$

$$T = A_1 A_2 A_3$$

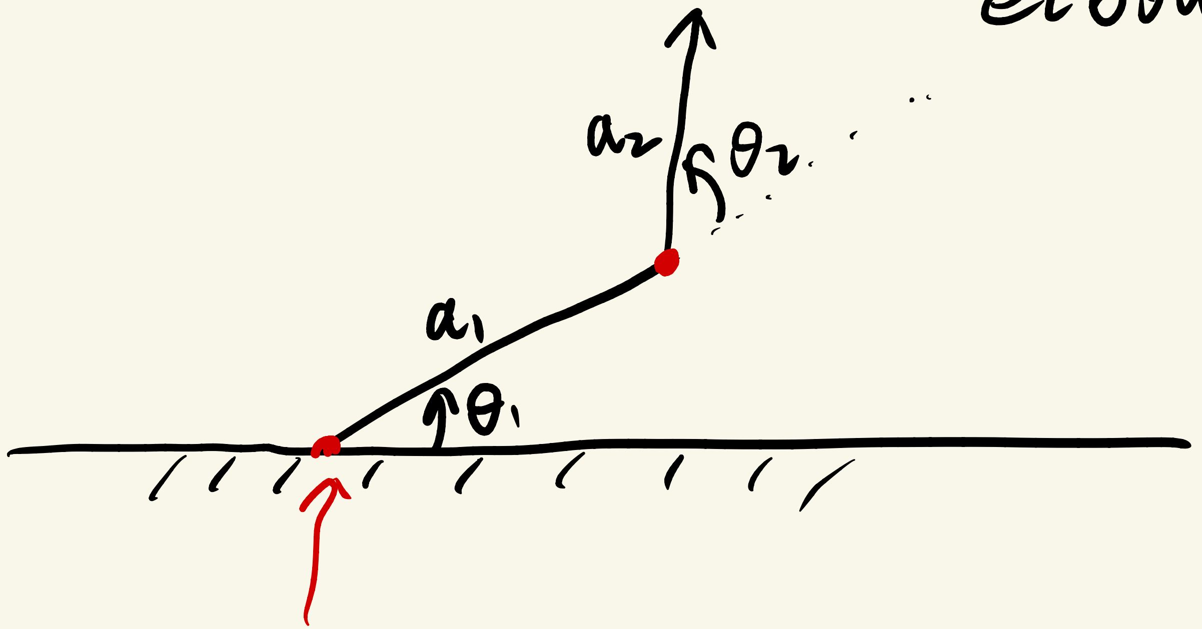
$$= \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

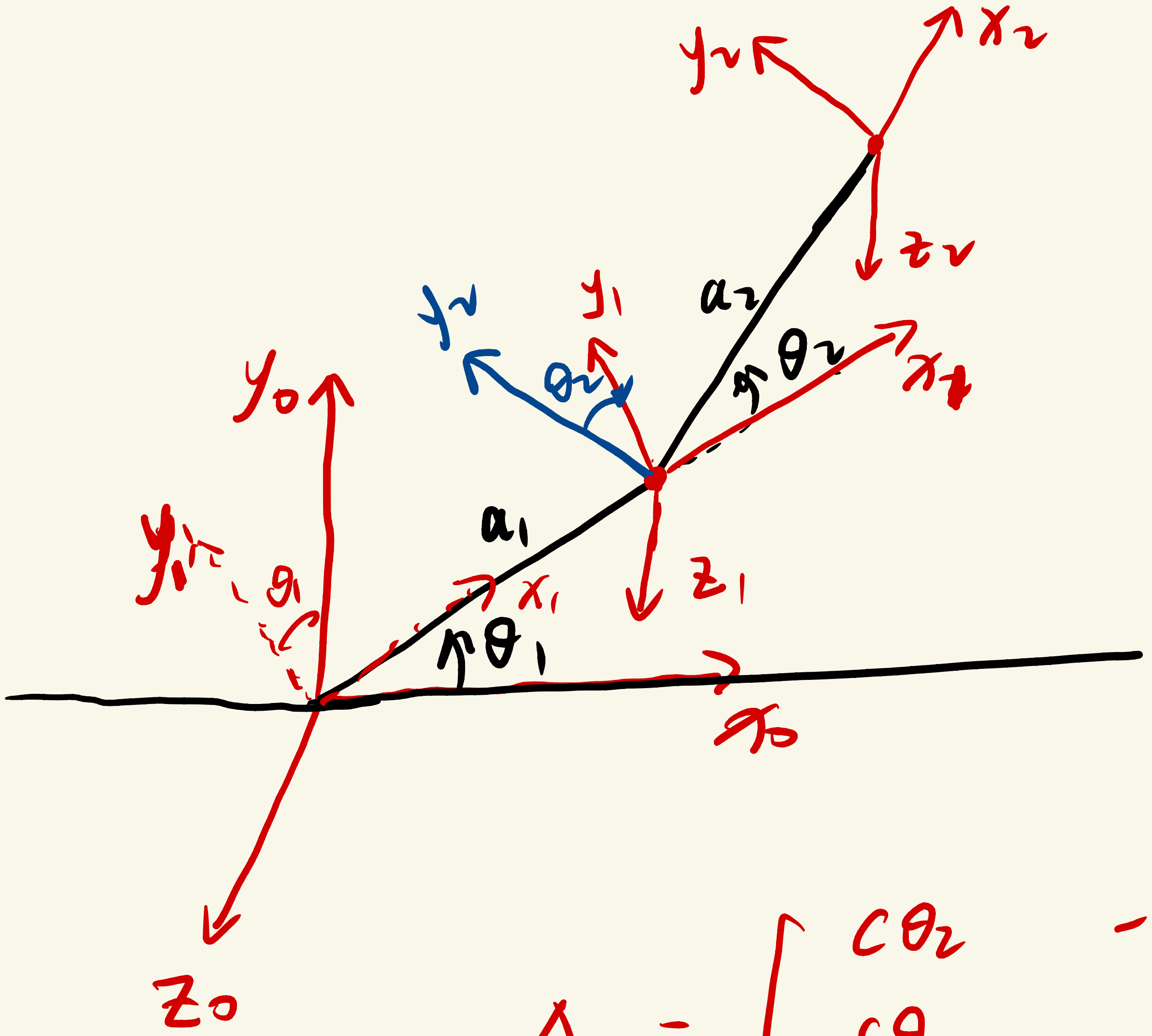
$$= \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
= \begin{array}{c}
\begin{array}{ccc|c} \quad & R & \quad & P \end{array} \\
\left[\begin{array}{ccc|c}
c\theta & -s\theta & 0 & rc\theta \\
s\theta & c\theta & 0 & rs\theta \\
0 & 0 & 1 & h \\
\hline
0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$$
P = [\, rc\theta \quad rs\theta \quad h \,]^T
$$

Example 2 kinematics for two-link planar elbow arm





$$A_1 = \left[\begin{array}{ccc|c} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$A_2 = \left[\begin{array}{ccc|c} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$c \triangleq \cos$

$s \triangleq \sin$

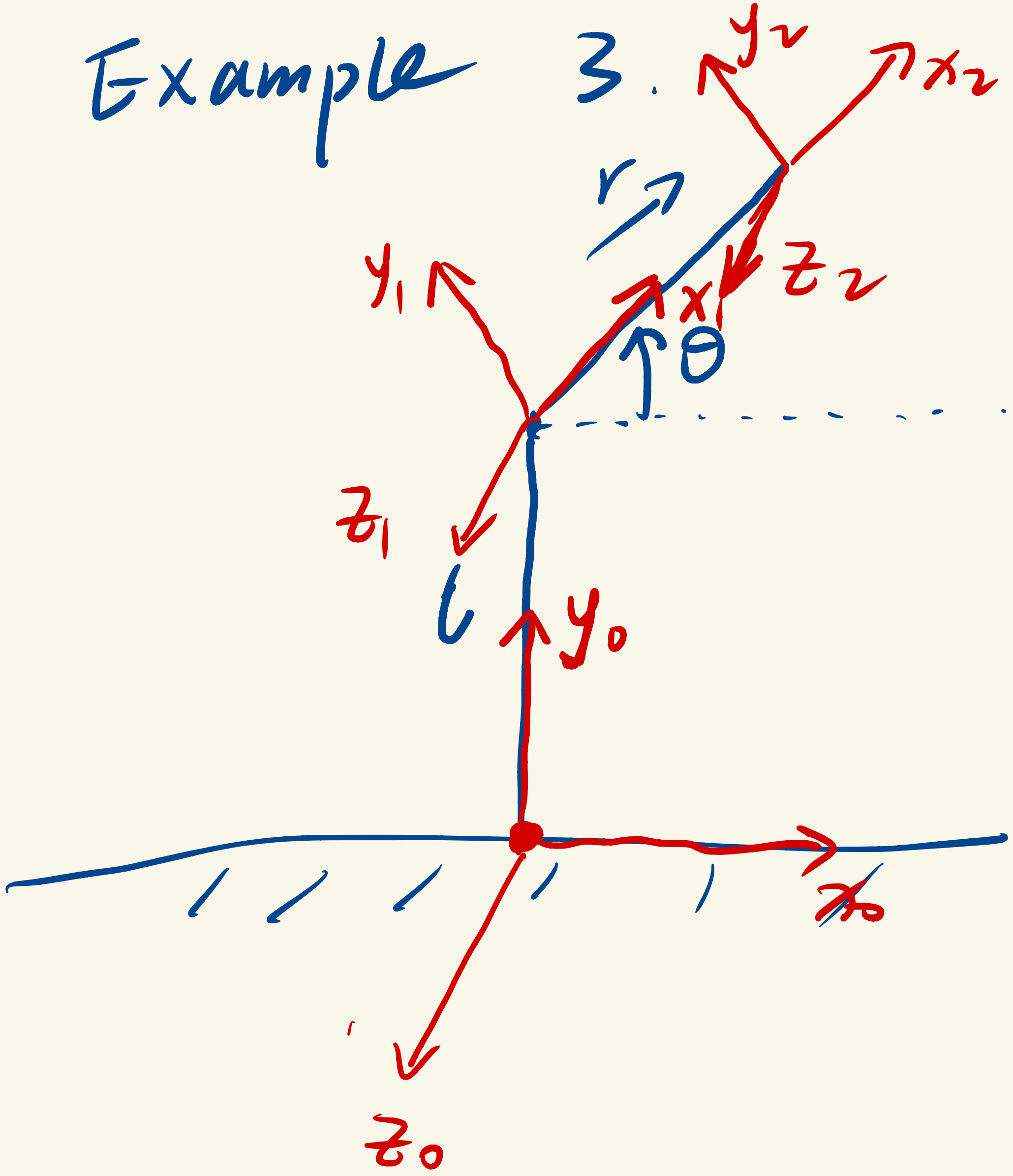
$$T = A_1 A_2$$

$$= \left[\begin{array}{ccc|c}
\overset{\cos(\theta_1+\theta_2)}{\boxed{c_1c_2 - s_1s_2}} & \overset{-\sin(\theta_1+\theta_2)}{\boxed{-c_1s_2 - c_2s_1}} & 0 & \overset{a_1c_1 + a_2(c_1c_2 - s_1s_2)}{\boxed{a_1c_1 + a_2 c(\theta_1+\theta_2)}} \\
\overset{\sin(\theta_1+\theta_2)}{\boxed{s_1s_2 + c_1c_2}} & \overset{\cos(\theta_1+\theta_2)}{\boxed{-s_1s_1 + c_1c_2}} & 0 & a_1s_1 + a_2 s(\theta_1+\theta_2) \\
0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 1
\end{array}\right]$$

$$\cos(\theta_1+\theta_2)$$

$$= \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

Example 3. Two-link polar arm



$q = [\, l \quad \theta \,]^T$ ✓

$$A_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = A_1 A_2 = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & r\cos\theta \\ \sin\theta & \cos\theta & 0 & l + r\sin\theta \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

translation vector

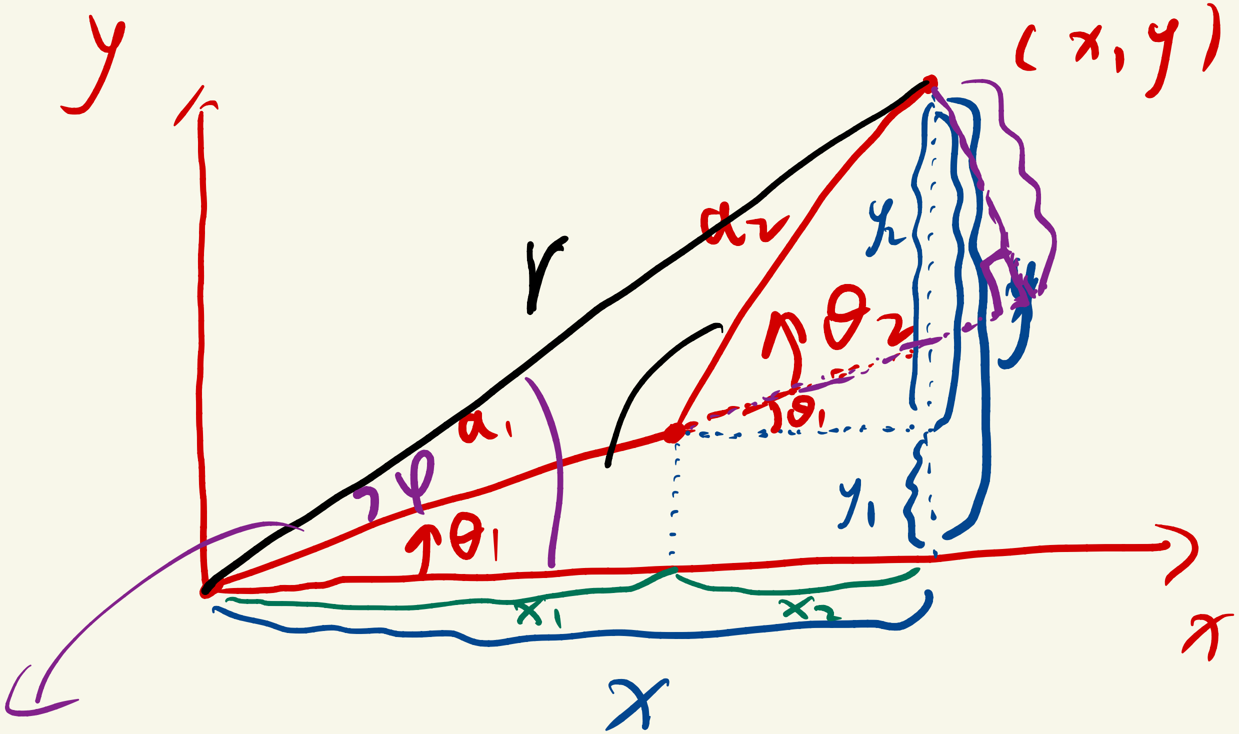
$$P = [\, r\cos\theta \quad l + r\sin\theta \quad 0 \,]^T$$

$$= \underline{\hspace{10em}}$$

Inverse Kinematics (IK)

The IK problem is given (n o a p) for the end-effector in the base framework, determine variable q_i in the T matrix.

Example: IK for Two-link Planar Elbow Arm



$$\tan\varphi = \frac{a_2 \sin\theta_2}{a_1 + a_2\cos\theta_2}$$

$$y = y_1 + y_2$$

$$y_1 = a_1 \sin\theta_1$$

$$y_2 = a_2 \sin(\theta_1+\theta_2)$$

$$y = a_1 \sin\theta_1 + a_2 \sin(\theta_1+\theta_2)$$

$$x = x_1 + x_2$$

$$= a_1 \cos\theta_1 + a_2 \cos(\theta_1+\theta_2)$$

To find θ_2,

$$r^2 = x^2 + y^2$$

use the law of cosins to obtain

$\cos\pi\cos\theta_2 + \sin\pi\sin\theta_2$

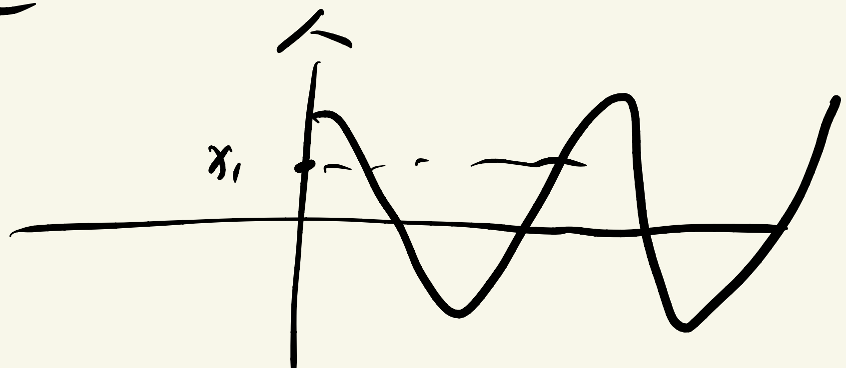
$$r^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \theta_2)$$

$$= a_1^2 + a_2^2 + 2a_1a_2\cos\theta_2$$

$$\cos\theta_2 = \frac{r^2 - a_1^2 - a_2^2}{2a_1a_2} \equiv C$$

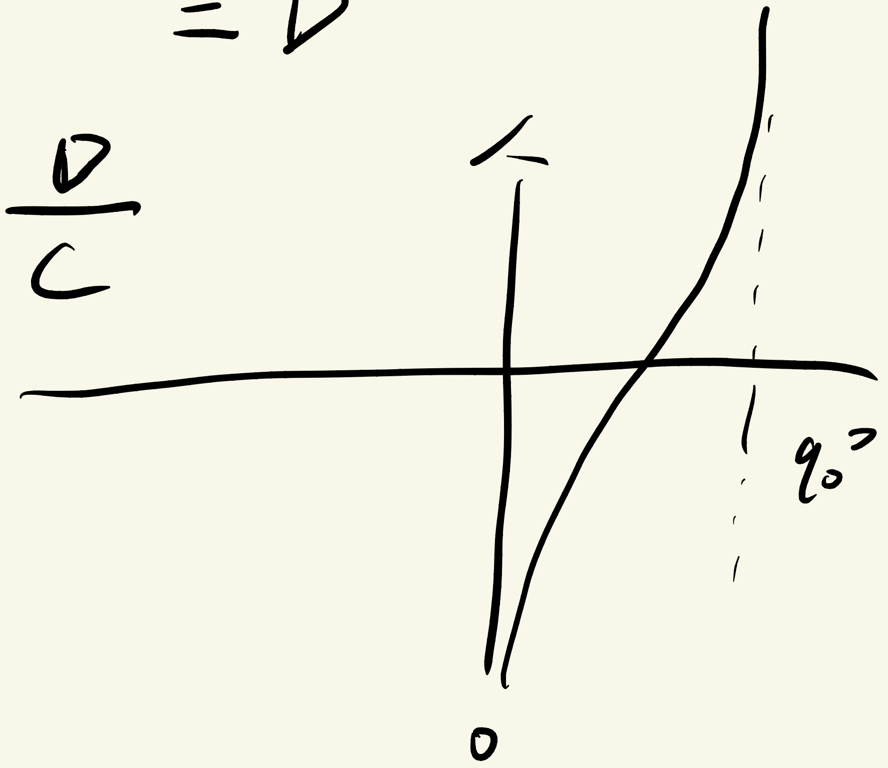
$$\theta_2 = \arccos C$$

yield nonunique solutions



$$\sin\theta_2 = \pm\sqrt{1-c^2} \equiv D$$

$$\tan\theta_2 = \frac{\sin\theta_2}{\cos\theta_2} = \frac{D}{C}$$



$$\theta_2 = \arctan\frac{D}{C}$$

To find θ_1

$$\tan\varphi = \frac{a_2\sin\theta_2}{a_1 + a_2\cos\theta_2}$$

$$\tan(\varphi + \theta_1) = \frac{y}{x}$$

$$\varphi + \theta_1 = \arctan\frac{y}{x}$$

$$\theta_1 = \arctan\frac{y}{x} - \arctan\frac{a_2 \sin\theta_2}{a_1 + a_2 \cos\theta_2}$$

The Manipulator Jacobian

Given a generally nonlinear transformation

from joint variable $q(t) \in R^n$ to $y(t) \in R^p$

$$y = h(q)$$

We define Jacobian associated with $h(q)$ as

$$J(q) = \frac{\partial h(q)}{\partial q}$$

I. Transformation of velocity and acceleration

II. Transformation of force

I. If y is position vector,

then the velocity is $\frac{dq}{dt}$

$$\frac{dy}{dt} = \frac{\partial h}{\partial q} \cdot \frac{\partial q}{\partial t} = J \dot{q}$$

$$\Downarrow$$

$$\dot{y} = \underline{J\,\dot{q}}$$

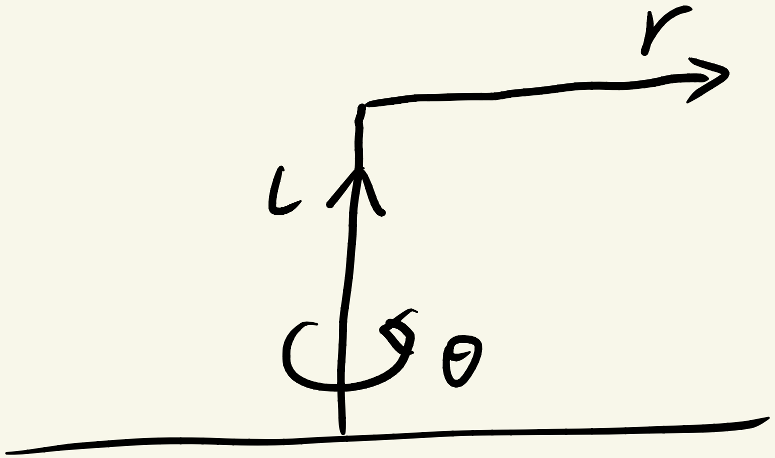
velocity in Cartesian space

velocity in joint space

Generally, $\dot{y} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{bmatrix}_{6\times 1}$

If q is $n \times 1$, then J is $6 \times n$

Example 1.



Translation vector of end effector is

$[\ -r\sin\theta \quad r\cos\theta \quad l\ ]^T$

Define
$$y_p = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ l \end{bmatrix}$$ as the position of end effector

$$\equiv h(q) \quad , \quad q = \begin{bmatrix} \theta \\ l \\ r \end{bmatrix}$$

$$J = \frac{\partial h}{\partial q}$$

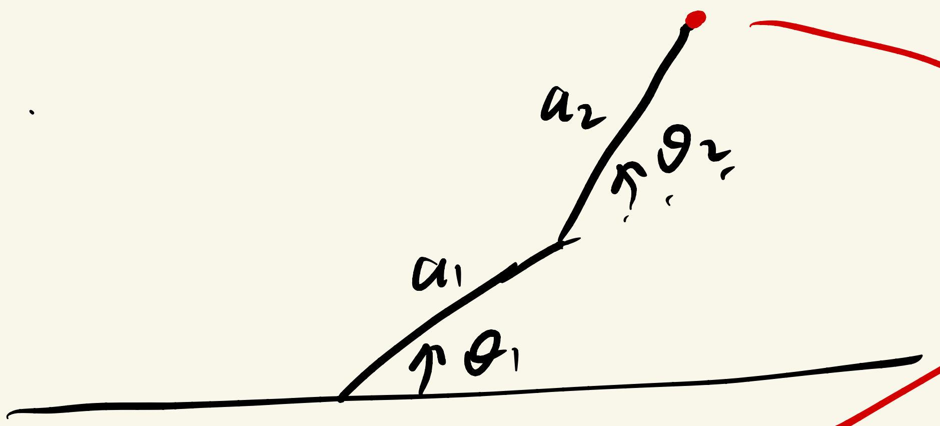
$$= \begin{bmatrix} \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial l} & \frac{\partial h}{\partial r} \end{bmatrix}$$

$$\begin{bmatrix} y_{r_{3\times1}} \\ 0 \\ 0 \\ 0 \end{bmatrix}_{6\times1} = \left[\frac{\partial h}{\partial q} \atop \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right]_{6\times3} \begin{bmatrix} \dot\theta \\ \dot l \\ \dot r \end{bmatrix}_{3\times1} = \begin{bmatrix} \frac{\partial(-r\sin\theta)}{\partial\theta} = -r\cos\theta & 0 & -\sin\theta \\ \frac{\partial(r\cos\theta)}{\partial\theta} = -r\sin\theta & 0 & \cos\theta \\ \frac{\partial(l)}{\partial\theta} = 0 & 1 & 0 \end{bmatrix}$$

$$\frac{dy_p}{dt}_{3\times1} = J_{3\times3}\,\frac{dq}{dt}_{3\times1}$$

$$\dot y_p = J\dot q = \begin{bmatrix} -r\cos\theta & 0 & -\sin\theta \\ -r\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta \\ \dot l \\ \dot r \end{bmatrix}$$

Example 2.



translation vector in base frame

$$y_p = \begin{bmatrix} a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \end{bmatrix}_{2\times 1} \equiv h(q)$$

$$q = [\theta_1 \quad \theta_2]^T$$

$$J = \frac{\partial h(q)}{\partial q} = \begin{bmatrix} \frac{\partial h}{\partial \theta_1} & \frac{\partial h}{\partial \theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1+\theta_2) & -a_2 \sin(\theta_1+\theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1+\theta_2) & a_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\dot{y}_P = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Transformation of Acceleration

$$\dot{y} = J\dot{q}$$

$$\frac{d\dot{y}}{dt} = \frac{dJ}{dt}\dot{q} + J\frac{d\dot{q}}{dt}$$

$$\ddot{y} = \dot{J}\dot{q} + J\ddot{q}$$

Cartesian acceleration

Joint acceleration

II. Transformation of force

The virtual work δW resulting from the application of a force F in Cartesian space that results in a linear motion dy

$$\delta W = F^T dy$$

$$= \tau^T dq \longrightarrow \text{rotation}$$

rotational force

Then, $F^T dy = \tau^T dq$

Because $\frac{dy}{dt} = J \frac{dq}{dt}$

$dy = J\,dq$

So, $F^T J\,dq = \tau^T dq$

$\tau = J^T F$ //