ELEC 5530/6630
Robotics
Lecture 2: State-Variable Systems and Simulation using Matlab

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❑ State-Variable Systems

- State-variable systems
- Differential equations to state-variable systems
- Examples

❑ Matlab for Solving State-Variable Systems

- Matrix operations and indexing
- Ode functions with examples

- State-variable systems / state space representation

A state-variable system models a dynamic system and comprises a first-order ordinary differential system (ODE) that describes time derivatives of a set of state variables.

The most general state-variable system of a linear system with *p* inputs, *q* outputs and *n* state variables is written in the following form

$\frac{dx}{dt}$ or x' $\leftarrow$

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

$x(t) \in \mathbb{R}^n$: state vector

$y(t) \in \mathbb{R}^q$: output vector

$u(t) \in \mathbb{R}^p$: control input vector

$A(t) \in \mathbb{R}^{n\times n}$: state matrix

$B(t) \in \mathbb{R}^{n\times p}$: input matrix

$C(t) \in \mathbb{R}^{q\times n}$: output matrix

$D(t) \in \mathbb{R}^{q\times p}$: feedforward matrix

Exp: $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ $\quad \dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = x_1(t) + u(t)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

If (A, B, C, D) are time invariant,

the syseem is time invariant.

Discrete-time state variable syseems

$$x(k+1) = A\,x(k) + B\,u(k)$$

$$y(k+1) = C\,x(k) + D\,u(k)$$

- Nonlinear state variable syseems

$$\begin{cases} \dot{x}(t) = f(x, u) \\ y(t) = h(x, u) \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sin x_1 \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin x_2 \end{bmatrix}$$

$$A(x,u) = \frac{\partial f}{\partial x}, \quad B(x,u) = \frac{\partial f}{\partial u}$$

$$C(x,u) = \frac{\partial h}{\partial x}, \quad D(x,u) = \frac{\partial h}{\partial u}$$

- Controllability: the ability to seeer states from any initial values to any final values by desinging some control inputs.
- Observability: the ability to infer internal states from exeernal outputs

How to check

- controllability :

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$$

- Observability :

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

- Ordinary differential equations to state-variable systems

$$m\frac{d^2x}{dt^2} = F(x(t))$$

Define $z_1 = x$

$z_2 = \dot{x}$

$$\begin{cases} \dot{z}_1 = \dot{x} = z_2 \\ \dot{z}_2 = \ddot{x} = \frac{F}{m} \end{cases}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{m} \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \frac{F}{m}$$

- Ordinary differential equations to state-variable systems

In general n-th ODE

$$\frac{d^n y}{dt^n} + \alpha_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + \alpha_1 \frac{dy}{dt} + \alpha_0 y = 0$$

step 1 : define new variables starting with y;

the No. of ↓ is the order of ODE

step 2 : express the derivative of new variables in terms of new variables

- Examples

Step1: define $z_1 = y$

$z_2 = \dot{y}$

$z_3 = \ddot{y}$

$\vdots$

$z_n = y^{(n-1)}$

step 2:

$\dot{z}_1 = z_2$

$\dot{z}_2 = z_3$

$\vdots$

$\dot{z}_n = \frac{d^n y}{dt^n}$

$= -\alpha_0 z_1 - \alpha_1 z_2 - \alpha_2 z_3 \cdots - \alpha_{n-1} z_n$

because $\frac{d^n y}{dt^n} = -\alpha_{n-1} \underbrace{\left(\frac{d^{n-1} y}{dt^{n-1}} \right)}_{\to z_n} \cdots - \alpha_1 \frac{dy}{dt} - \alpha_0 y$

$= -\alpha_{n-1} z_n \quad - \alpha_1 z_2 - \alpha_0 z_1$

- Examples

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \vdots \\ \dot{z}_n = -\alpha_0 z_1 - \alpha_1 z_2 - \alpha_2 z_3 \cdots - \alpha_{n-1} z_n \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

- Examples

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 0$$

Define
$$x_1 = y$$
$$x_2 = \dot{y}$$

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = -\dot{y} - y = -x_2 - x_1$$

- Examples

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = A x$$

- Examples $\ddot{y} + y^2\dot{y} + y = 0$

$$x_1 \triangleq y$$

$$x_2 = \dot{y}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -y^2\dot{y} - y = -x_1^2 x_2 - x_1$$

- Examples

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1^2 x_2 - x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1^2 x_2 - x_1 \end{bmatrix}$$

$$\dot{x} = f(x)$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- Examples

2 degree of freedom robot arm model

$$M\ddot{q}(t) + ML\dot{q}^2(t) + MgL\sin q(t) = u(t)$$

M : mass

L : length of link

g : gravity term

u : force

q : joint variables

- Examples

Define z_1 z_2

$$z_1 = q(t)$$

$$z_2 = \dot{q}(t)$$

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -\left(L z_2^2 + gL \sin z_1\right) + u(t)/M$$

- Examples

$$\dot{z} = f(z) + g(z)u$$

$$\Downarrow$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -Lz_2^2 - gL\sin z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$