ELEC 3500
Control Systems
Lecture 12: Root Locus Method

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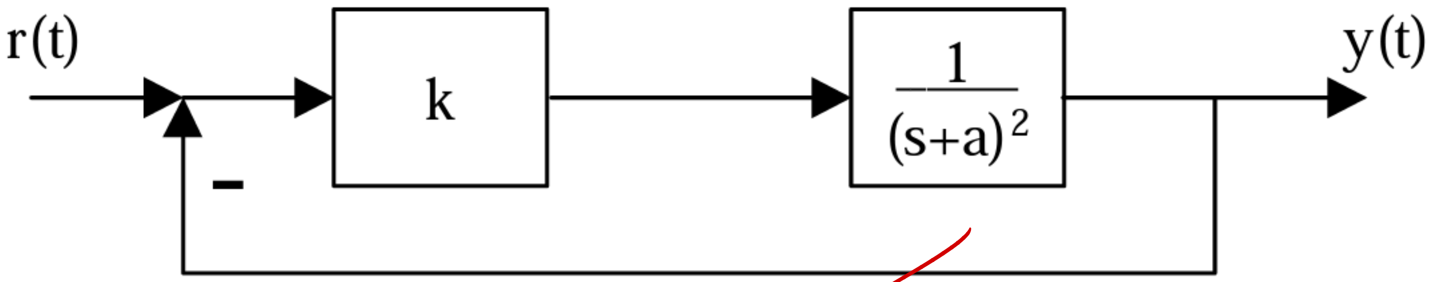
Variation of Closed-Loop Poles vs. a Design Gain

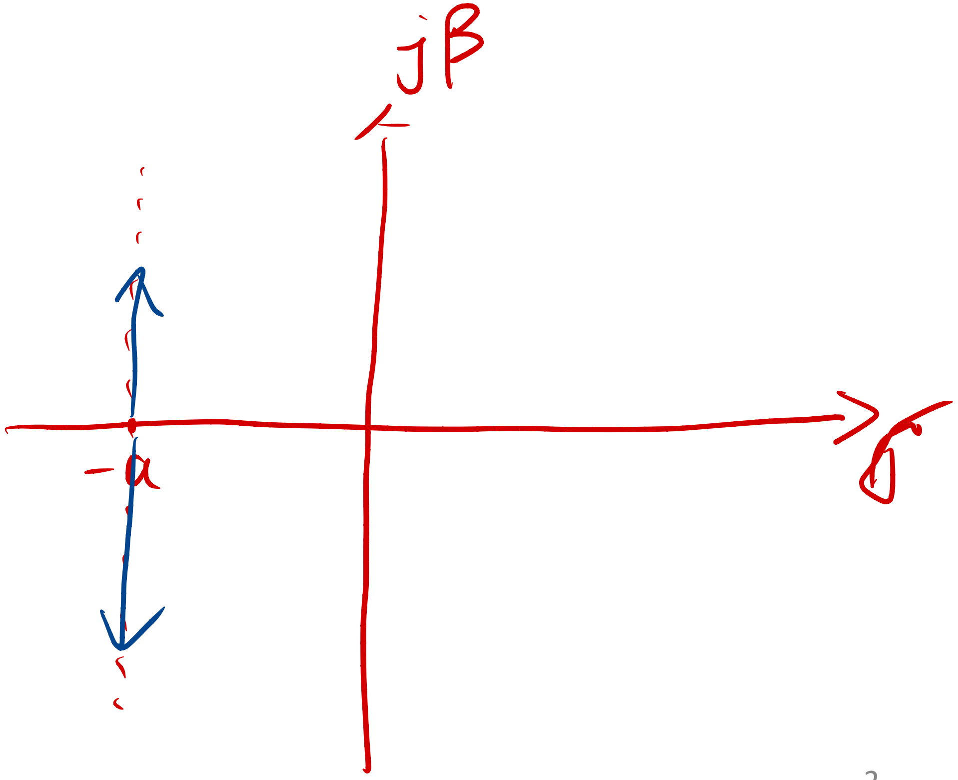
$G(s)$



$$\frac{kG(s)}{1+kG(s)}$$

The closed-loop transfer function is

$$T(s)=\frac{Y(s)}{R(s)}=\frac{\dfrac{k}{(s+a)^2}}{1+\left(\dfrac{k}{(s+a)^2}\right)}=\frac{k}{(s+a)^2+k}.$$



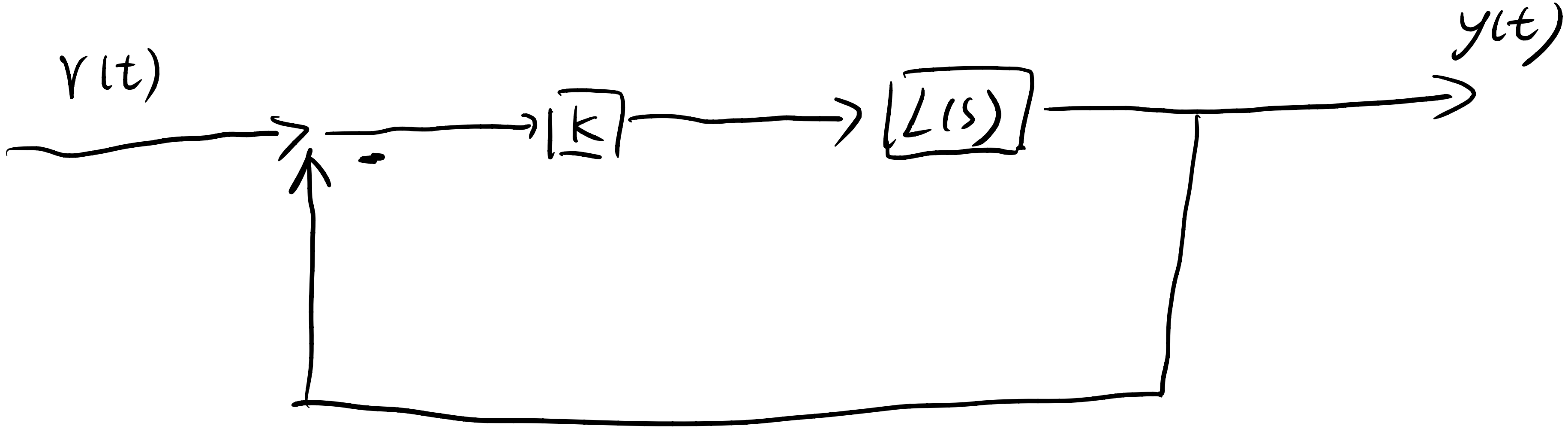
if $k>0$, $a>0$,

$-a+j\sqrt{k}$

$-a-j\sqrt{k}$

How the poles vary with *k* without ever finding the actual roots?

Root Locus Method



closed-loop transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K \cdot L(s)}{1 + K L(s)}$$

$$1 + K \cdot \frac{(\quad L \cdot \quad)}{L}$$

how poles vary as K increases from 0 to $+\infty$

The root locus for $1+KL(s)$ is the set of all closed-loop poles when K increases from 0 to ∞

roots of $1+KL(s)=0$

Define $L(s) \triangleq \dfrac{b(s)}{a(s)}$

Rules for Sketching Root Loci (Rules A — F)

highest order of a(s)

•Rule A — number of branches (= deg(a(s))) → highest order of a(s)

•Rule B — start points (= poles of $L(s)$) → roots of $a(s)=0$

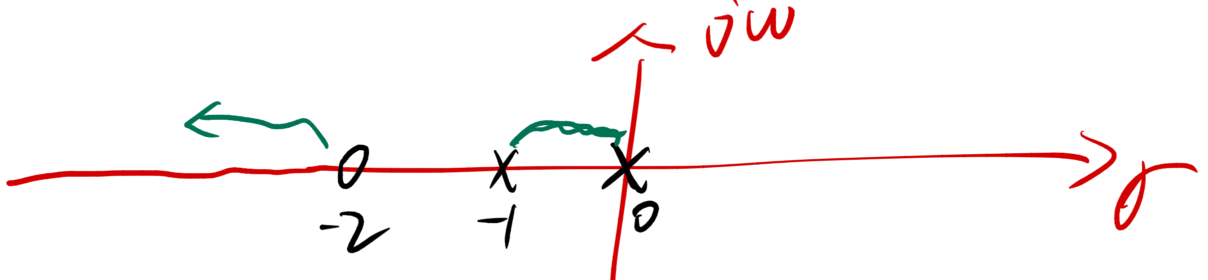
•Rule C — end points (= zeros of $L(s)$) → roots of $b(s)=0$

•Rule D — real locus (located relative to **real** open-loop poles/zeros) (This angles sum must be ±180° for **any** s that lies on the root locus.)

→ the real axis to the left of an odd numbers of poles and zeros is on the real locus

•Rule E — asymptotes

•Rule F — jω-crossings



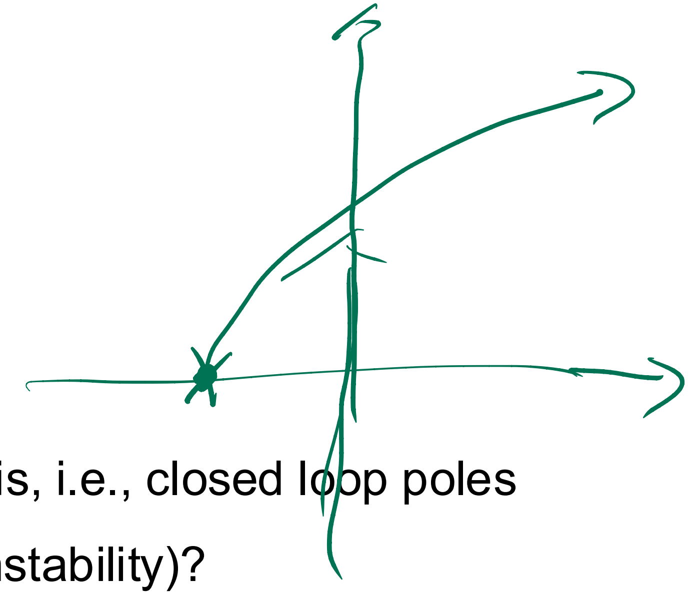
Rule E — asymptotes

When $s \to \infty$, how does the locus look like at infinity?

Rule E: Branches of root locus at ∞ have phase

$$\angle s \approx \frac{180^\circ + \ell \cdot 360^\circ}{n-m} = \frac{(2\ell+1)\cdot 180^\circ}{n-m}, \qquad \ell = 0, 1, \ldots, n-m-1.$$

n is deg $(a(s))$
m is deg $(b(s))$



Rule F — $j\omega$-crossings

Do the branches of the root locus cross the imaginary axis $j\omega$-axis, i.e., closed loop poles transitioned from Left Half Plane (stability) to Right Half Plane (instability)?

$$a(j\omega) + Kb(j\omega) = 0.$$

Use the **Routh test** to first determine the critical value of ***K*** when the characteristic polynomial is about to become unstable. Then plug this ***K*** back into the above equation to solve for $j\omega$crossings.

Example 1

Plot root locus for

$$L(s) = \frac{s+1}{s(s+2)((s+1)^2+1)} \triangleq \frac{b(s)}{a(s)}$$

$b(s) = s+1$

$a(s) = s(s+2)\left((s+1)^2+1\right)$

Rule A: branches:

$\deg(a(s)) = 4$

Rule B: start points:

$a(s) = 0$

poles: 0, -2, $-1+j$, $-1-j$

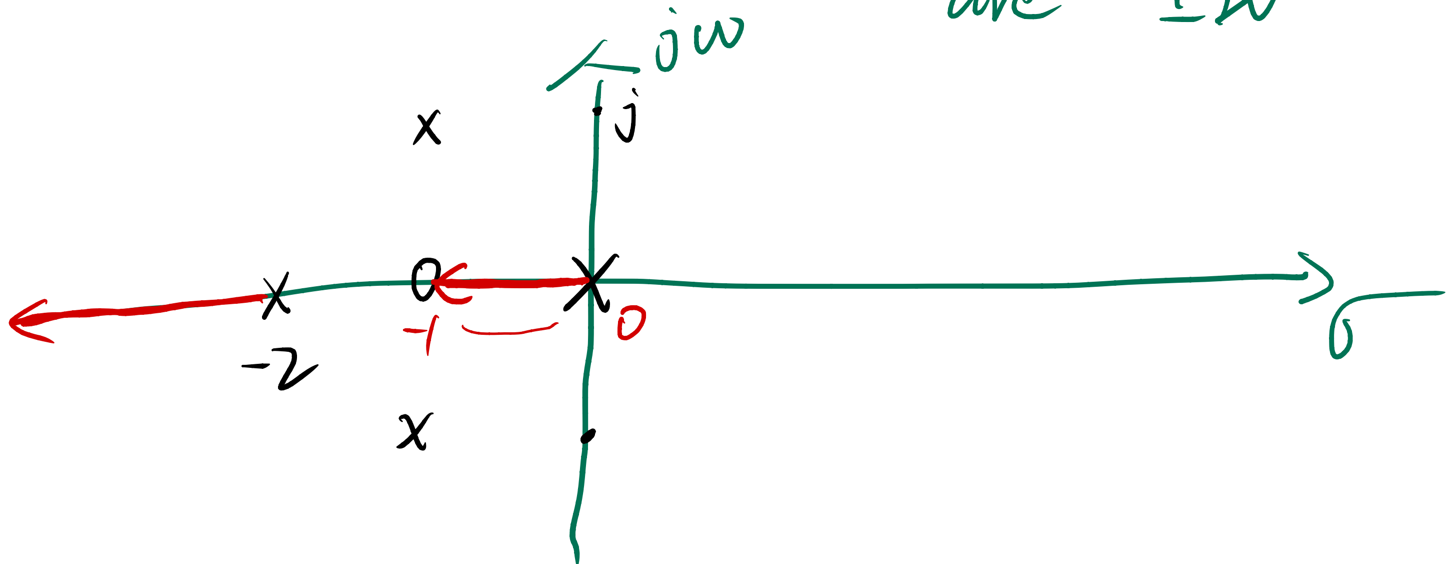
Rule C: end points

$b(s) = 0$

$s+1 = 0$

zeros $s=-1$, the other three end point zeros are $\pm\infty$

Rule D:



$$\angle s = \frac{180^\circ + l \cdot 360^\circ}{n-m}$$

Rule E:

$$l = 0,\ 1,\ \cdots,\ n-m-1$$

$$n = \deg(a(s)) = 4$$

$$m = \deg(b(s)) = 1$$

$4 \quad 1$

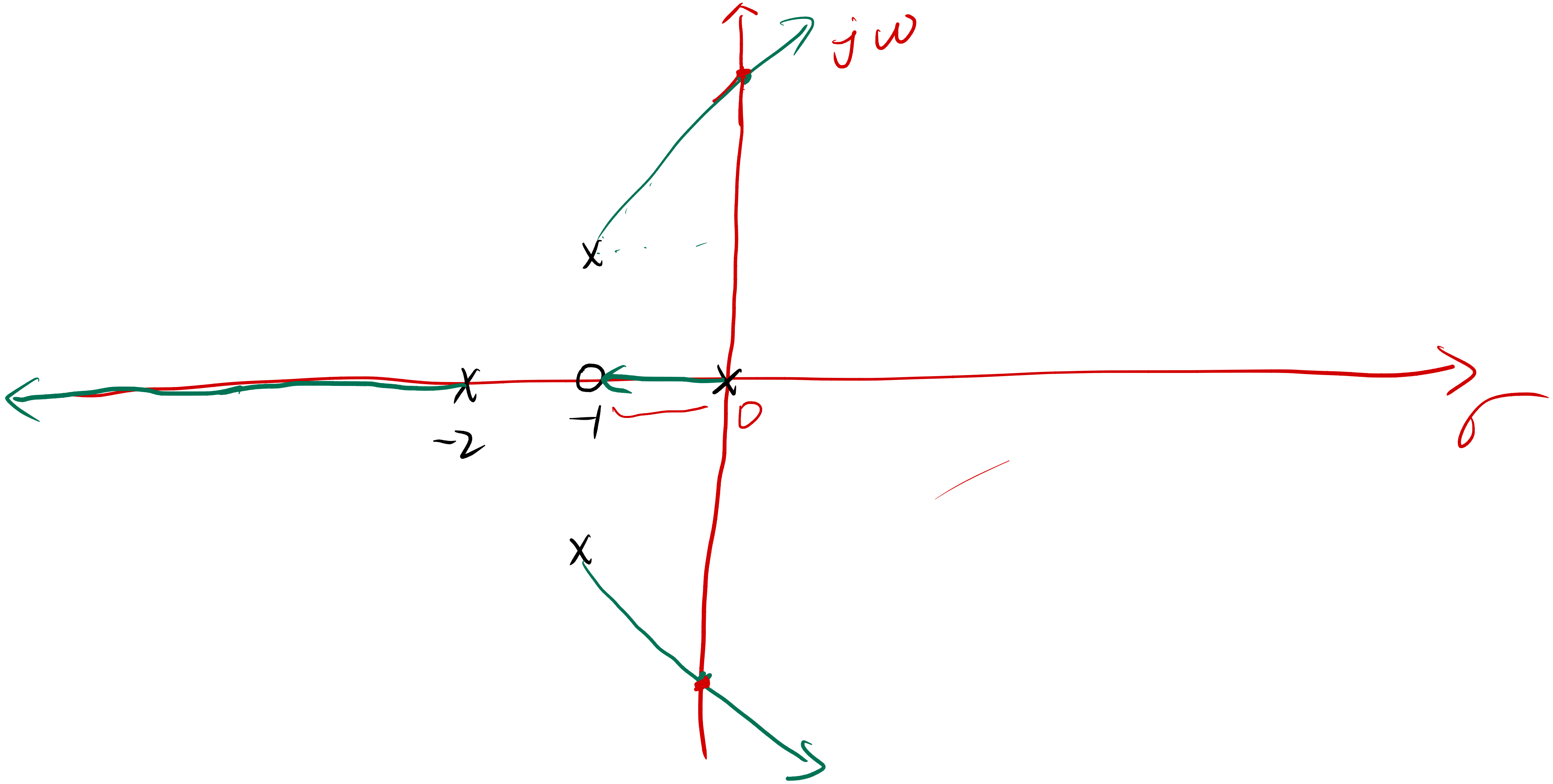
$n-m-1 = 2$

so, $l = 0,\ 1,\ 2$

$$\angle s_1 = \frac{180^\circ + 0 \cdot 360^\circ}{3} = 60^\circ$$

$$\angle s_2 = \frac{180 + 1 \cdot 360}{3} = 180^\circ$$

$$\angle s_3 = \frac{180 + 2 \cdot 360}{3} = 300^\circ = -60^\circ$$


jw
-2
-1
0
σ

Rule F.

$$a(s) + k \cdot b(s) = 0$$

$$s(s+2)\left((s+1)^2+1\right) + k \cdot (s+1) = 0$$

$$s^4 + 4 \cdot s^3 + 6s^2 + (4+k)s + k = 0$$

Routh Test:

s^4	1	6	K
s^3	4	$4+K$	
s^2	$C_1 = \frac{20-K}{4}$	K	
s	$\frac{80-K^2}{20-K}$		
s^0	K		

$$\begin{cases} 20-K > 0 \\ \frac{80-K^2}{20-K} > 0 \\ K > 0 \end{cases}$$

$$\Rightarrow 0 < K < 4\sqrt{5}$$

Critical value of K that makes system become unstable

$$a(j\omega) + 4\sqrt{5}\, b(j\omega) = 0$$

$$(j\omega)^4 + 4(j\omega)^3 + 6(j\omega)^2 + (4+4\sqrt{5})j\omega + 4\sqrt{5} = 0$$

$$\omega^4 + (-4j\omega^3) - 6\omega^2 + (4+4\sqrt{5})j\omega + 4\sqrt{5} = 0$$

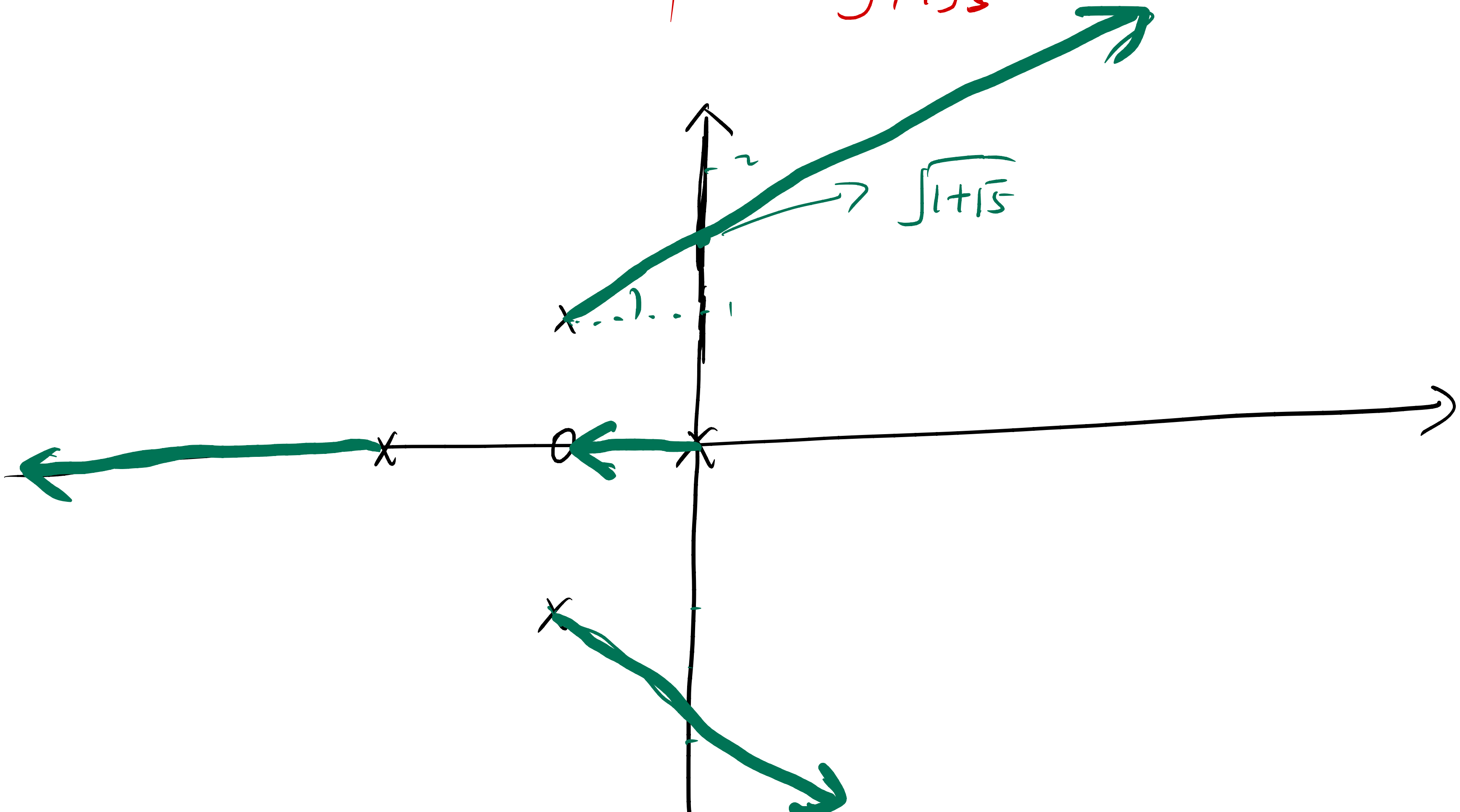
Real part $= 0 \rightarrow \omega^4 - 6\omega^2 + 4\sqrt{5} = 0$

Imaginary part $= 0 \rightarrow -4\omega^3 + 4+4\sqrt{5}\,\omega = 0$

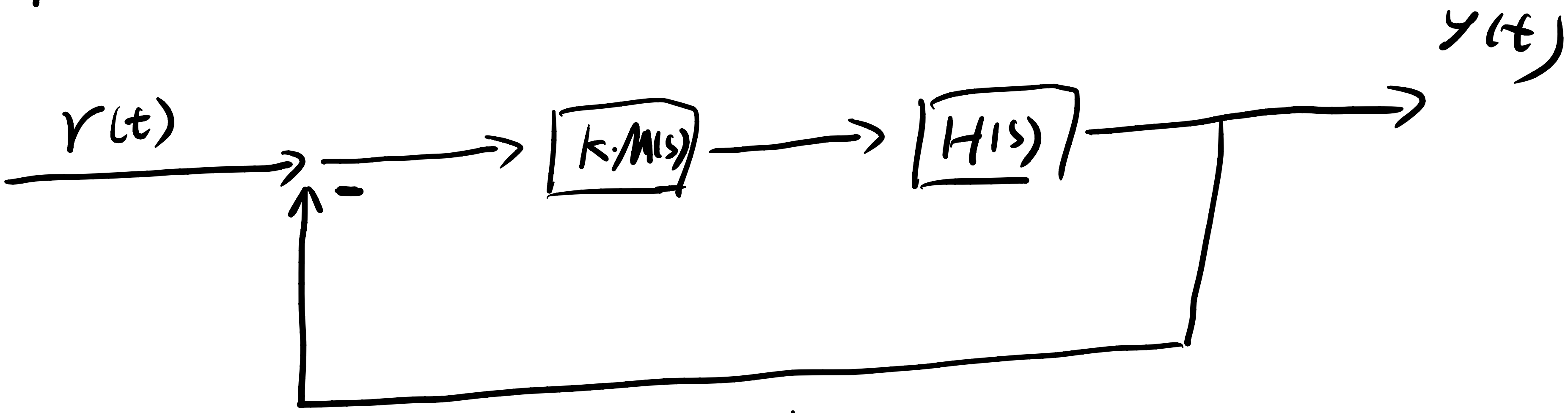
$$\omega^2 = 1+\sqrt{5}$$

$$\omega_0 = \pm\sqrt{1+\sqrt{5}}$$

$$1 < \sqrt{1+\sqrt{5}} < 2$$



Example 2.



set $H(s) = \dfrac{1}{(s+4)(s^2+2s+5)}$

$M(s) = \dfrac{1}{s}$

Closed-loop transfer function

$$T(s) = \frac{K \cdot M(s)\, H(s)}{1 + K M(s) H(s)}$$

$$= \frac{K M(s) H(s)}{1 + K \cdot L(s)}$$

where $L(s) = M(s) \cdot H(s)$

$$L(s) = \frac{1}{s(s+4)(s^2+2s+5)} \triangleq \frac{b(s)}{a(s)}$$

Rule A: branches

$\deg(a(s)) = 4$

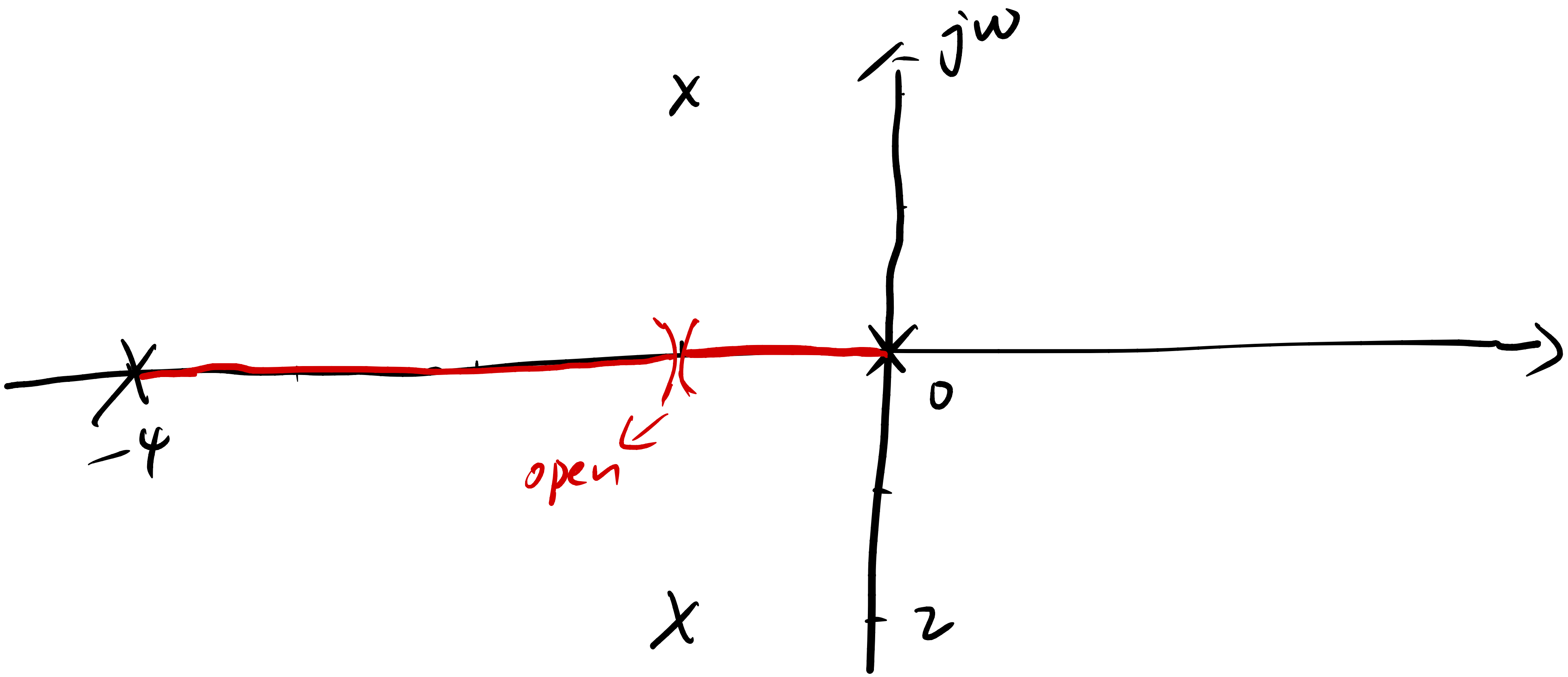
Rule B: start points : poles

$0, \quad -4, \quad -1 \pm 2j$

Rule C: end points : zeros

$\pm\infty$ (four)

Rule D: real locus



Rule E : phases of branches

$$\angle s = \frac{180^\circ + l \cdot 360^\circ}{n-m}$$

$n = 4$

$m = 0$

$l = 0, 1, 2, 3$

$\angle s_1 = 45^\circ$

$\angle s_2 = 135^\circ$

$\angle s_3 = 225^\circ$ $\angle s_4 = 315^\circ$

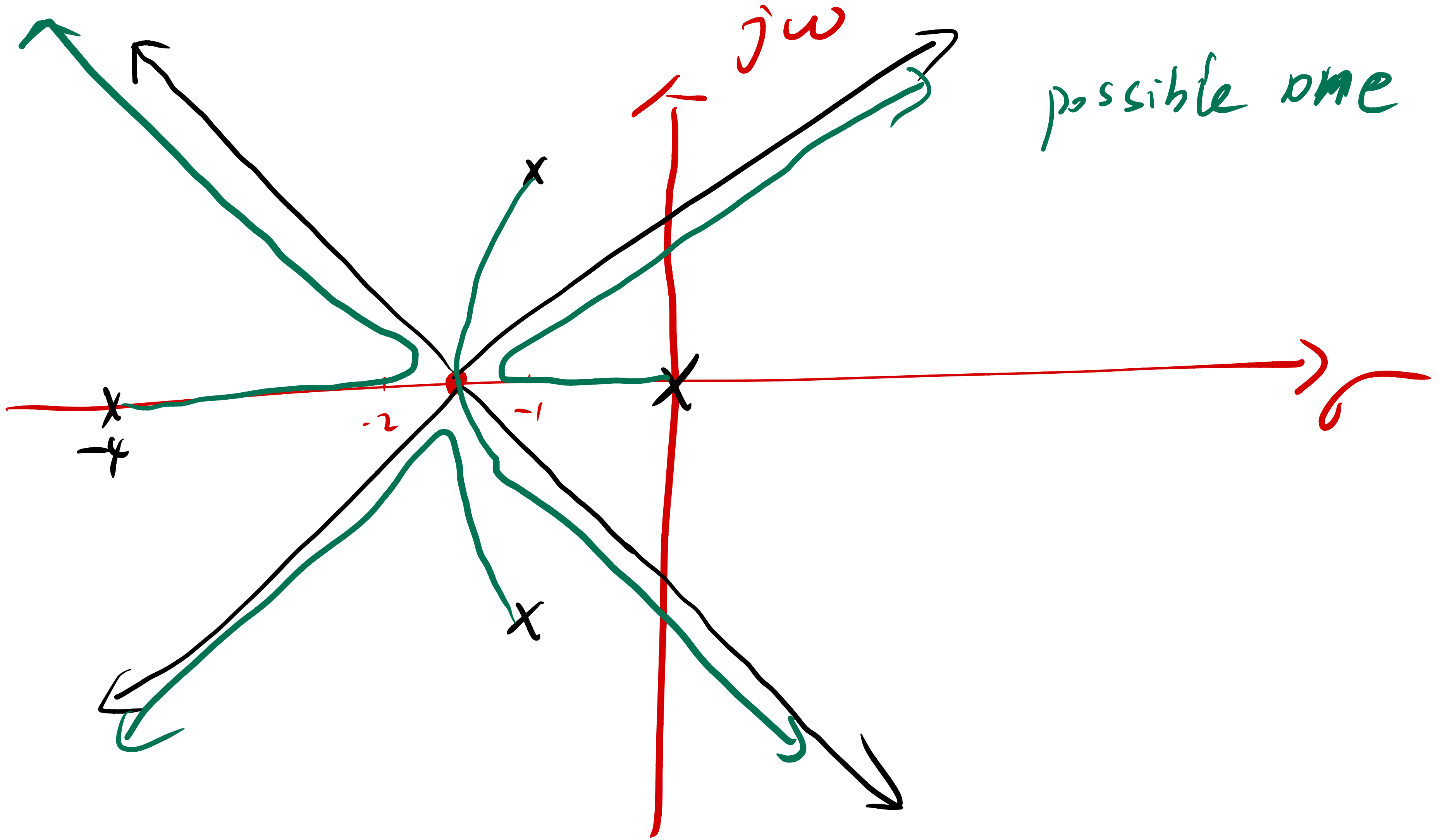
Centroid of asymptotes

— because there is no finite zeros

$$c = \frac{\Sigma(\text{poles}) - \Sigma(\text{zeros})}{n-m}$$

$$= \frac{0 - 4 + (-1+2j) - 1 - 2j - 0}{4}$$

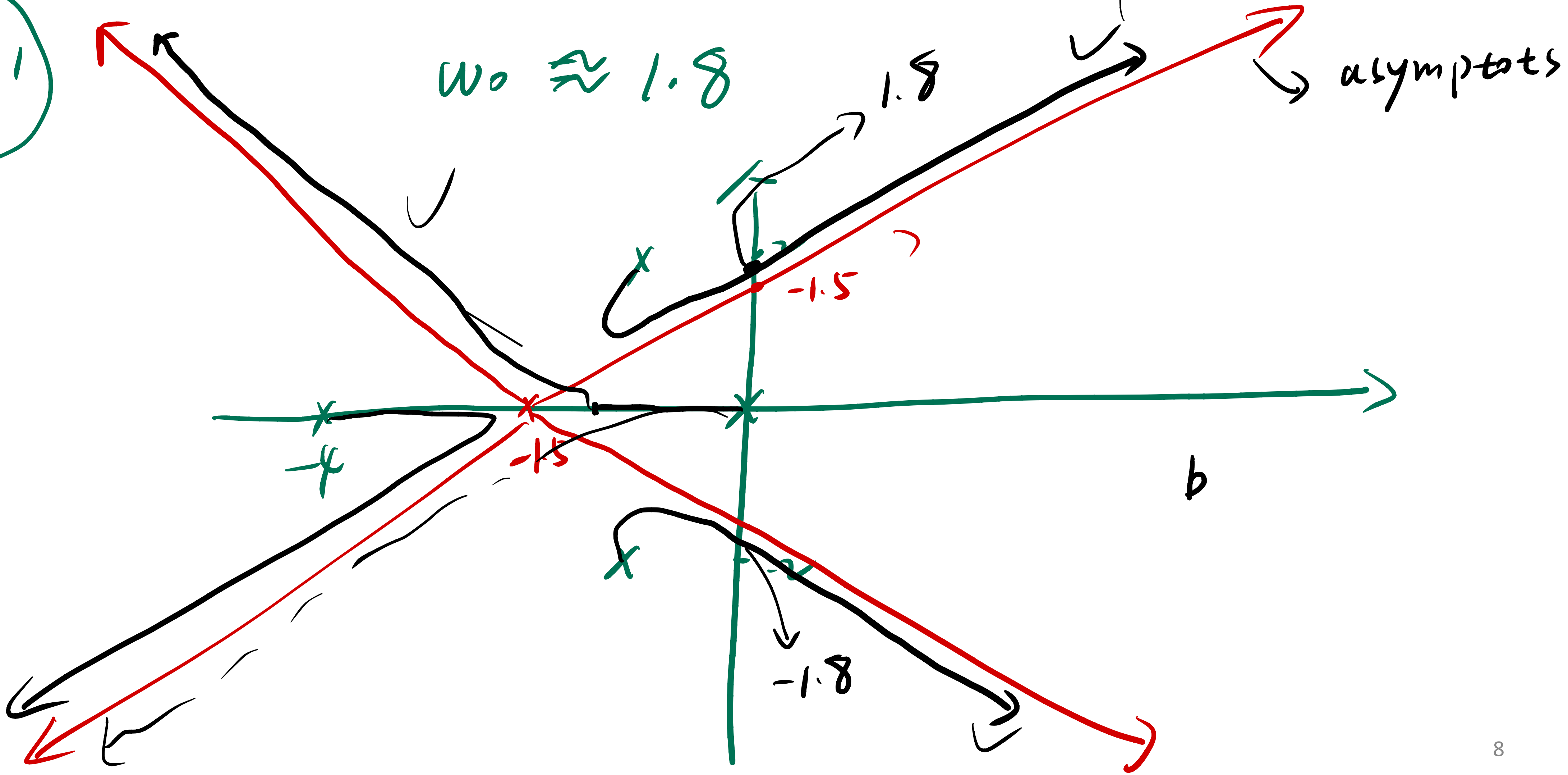
$$= -\frac{3}{2}$$


jω
possible one
σ
-4
-2
-1

Rule F $j\omega$-crossings

Solution 1

$\omega_0 \approx 1.8$



Solution 2