ROOT LOCUS DESIGN

Feedback control design techniques include state-variable feedback, output-feedback, root locus, Bode design, Nyquist design, polynomial design, and other techniques. Controls design techniques can be basically divided into state-space techniques vs. input/output (transfer function) techniques. They may also be classified as frequency domain vs. time domain techniques.

In this lecture we shall introduce the Root Locus technique, developed by Evans during World War 2 while working at North American Aviation. This is a highly intuitive pictorial technique for control system design working in the *s*-plane. It is an input/output, frequency domain design technique.

MATLAB is used for analysis and design. Many of the routines needed are in the MATLAB Control Systems Toolbox. A tutorial on MATLAB is available on the www at www.mathworks.com

Computing the Magnitude and Phase of a Complex Rational Function

Many design techniques for feedback control systems are based on frequency-domain notions. Therefore, we shall review the computation of the magnitude and phase of a complex rational function $G(s)$.

Recall that a complex number $z=a+jb$ has magnitude and phase given by

$$|z| = \sqrt{a^2 + b^2}$$

$$\angle z = \tan^{-1}\frac{b}{a} \quad .$$

The product of complex numbers

$$z = z_1 z_2 = (a + jb)(c + jd)$$

has

$$\begin{aligned} |z| &= |z_1| \cdot |z_2| = \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\ \angle z &= \angle z_1 + \angle z_2 = \tan^{-1}\frac{b}{a} + \tan^{-1}\frac{c}{d} \end{aligned}.$$

When dividing complex numbers

$$z = \frac{z_1}{z_2} = \frac{(a + jb)}{(c + jd)}$$

one has

$$\begin{aligned} |z| &= \frac{|z_1|}{|z_2|} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} \\ \angle z &= \angle z_1 - \angle z_2 = \tan^{-1}\frac{b}{a} - \tan^{-1}\frac{c}{d} \end{aligned}.$$

These facts allow us to easily compute the magnitude and phase of a prescribed function *G(s)* at a given value of *s*. The next example illustrates the very important connections between vectors drawn in the *s*-plane and computations regarding *G(s)*.

Example 1- Computing Magnitude and Phase of *G(s)* Using Vectors

Compute the magnitude and phase of

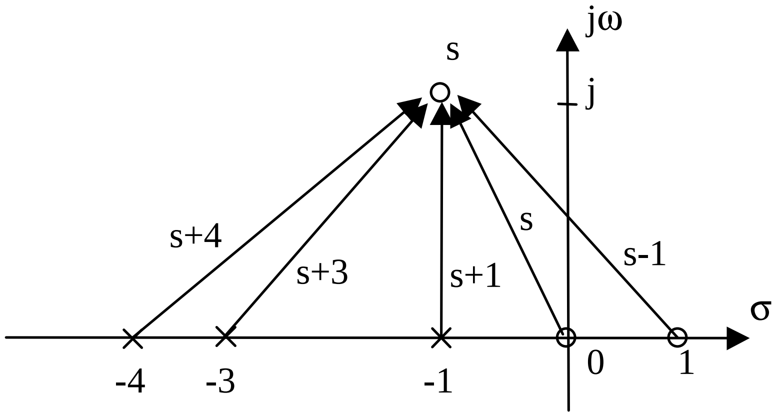
$$G(s) = \frac{s(s-1)}{(s+1)(s+3)(s+4)}.$$

at *s=-1+j*.

It is easy to solve this problem by hand by drawing vectors in the *s*-plane and working with their lengths and angles. As shown in the figure, to find magnitude and phase of *G(s)* at *s=-1+j* one simply plots the poles and zeros in the *s*-plane, and then draws a vector from each pole and zero to the prescribed point *s=-1+j*.

Note that each factor in *G(s)* has an associated vector in the *s*-plane. There are two sorts of vectors- pole vectors and zero vectors. It is easy to find the magnitudes and angles of these vectors. For instance, for the vector associated with the factor *(s+3)*, one sets *s=-1+j* and computes

$$\begin{aligned} |s+3| &= |2 + j| = \sqrt{2^2 + 1^2} = \sqrt{5} \\ \angle(s+3) &= \angle(2 + j) = \tan^{-1}\frac{1}{2} = 26.57^o \end{aligned}.$$



To find the magnitude of *G(s)* at *s=-1+j*, one divides the product of the magnitudes of the zero vectors by the product of the magnitudes of the pole vectors. The result is

$$|G(s)| = \frac{|s|\cdot|s-1|}{|s+1|\cdot|s+3|\cdot|s+4|} = \frac{|-1+j|\cdot|-2+j|}{|j|\cdot|2+j|\cdot|3+j|} = \frac{\sqrt{2}\sqrt{5}}{1\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{5}} = 0.447$$

To find the angle of *G(s)* at *s=-1+j*, one takes the sum of the angles of the zero vectors and subtracts the sum of the angles of the pole vectors. The result is

$$\begin{aligned}\angle G(s) &= \angle s + \angle(s-1) - \angle(s+1) - \angle(s+3) - \angle(s+4)\\ &= \angle(-1+j) + \angle(-2+j) - \angle(j) - \angle(2+j) - \angle(3+j)\\ &= \tan^{-1}\frac{1}{-1} + \tan^{-1}\frac{1}{-2} + \tan^{-1}\frac{1}{0} + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} \quad .\\ &= 135^o + 153.4^o - 90^o - 26.6^o - 18.4^o = 153.4^o\end{aligned}$$

Example 2- Plotting Magnitude and Phase of *G(s)* Using MATLAB

It is easy to plot the magnitude and phase of a given rational function *G(s)* using MATLAB.

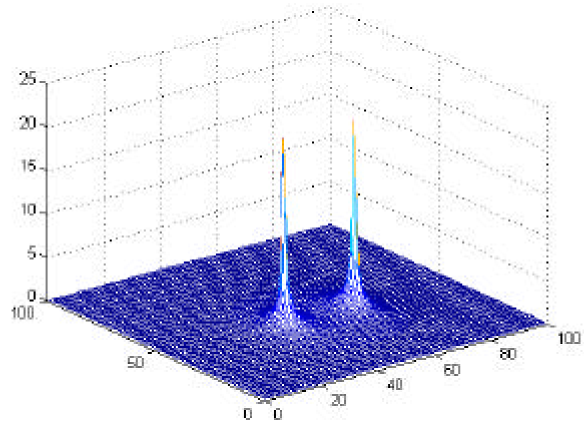
Let

$$G(s) = \frac{1}{(s+1)(s+3)}.$$

The magnitude of *G(s)* as a function of *s* is a surface over the *s*-plane. To plot this surface one uses the MATLAB commands:

```
sig=linspace(-6,2,100);
omeg=linspace(-2,3,100);
[Sig,Omeg]=meshgrid(sig,omeg);
s=Sig+Omeg*i;
den=abs(s+1).*abs(s+3);
num=ones(size(den));
z=num./den;
mesh(z)
```

The result is shown in the figure.

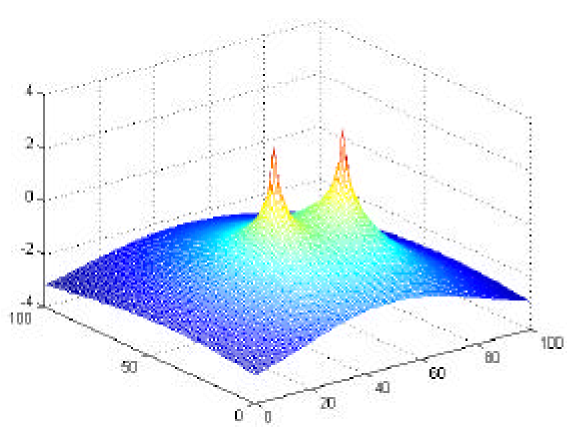


In the MATLAB command lines, note the use of the `linspace` command to generate linearly spaced points in sigma and j*omega. The `meshgrid` command is then used to produce two matrices as specified in the MATLAB Reference Guide. The matrix complex variable *s* is produced using the imaginary variable *i*. The function `abs` produces the element-by-element absolute value of a complex matrix. The variable `den` is a matrix produced by element-by-element matrix operations by placing the dot before the multiplication operator *. The `ones` command produces a matrix of 1's the same size as den. Finally, *z* is a matrix produced by element-by-element division by placing the dot prior to the division operator /.

One may alternatively plot the logarithm of the magnitude. To do so, one inserts the MATLAB command

```
z=log(z);
```

prior to the plot command mesh(z). The result is shown in the figure.



Example 3- Plotting Magnitude and Phase of *G(s)* Using MATLAB

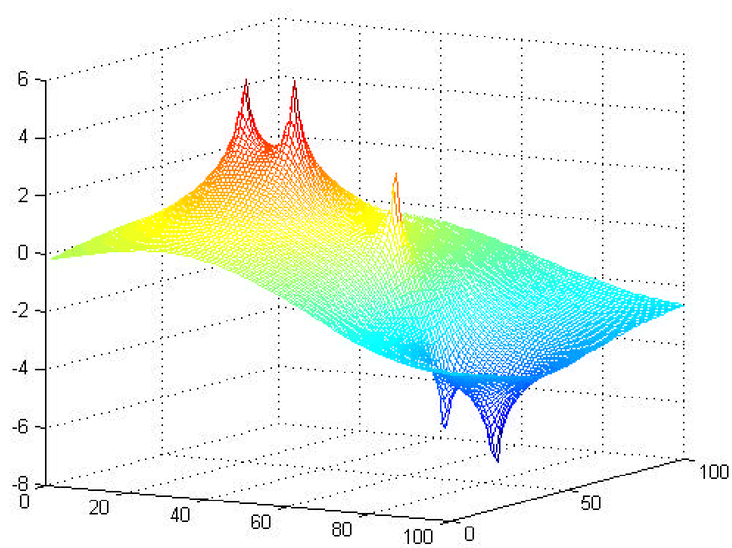
Let

$$G(s) = \frac{s(s-1)}{(s+1)(s+3)(s+4)}.$$

To plot the magnitude one uses the MATLAB commands:

```
sig=linspace(-6,2,100);
omeg=linspace(-2,3,100);
[Sig,Omeg]=meshgrid(sig,omeg);
s=Sig+Omeg*i;
num=abs(s).*abs(s-1);
den=abs(s+1).*abs(s+3).*abs(s+4);
z=num./den;
z=log(z);
mesh(z)
view(30,10) %view from azimuth=30 deg, elevation=10 deg
```

The result is the figure shown.



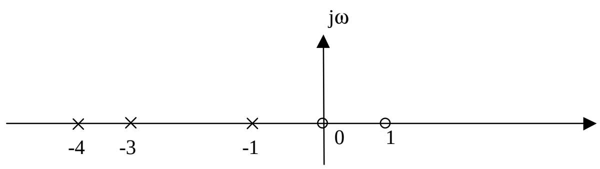
Phase of *G(s)* on the Real Axis

All complex rational functions have an important special property when the phase is evaluated on the real axis.

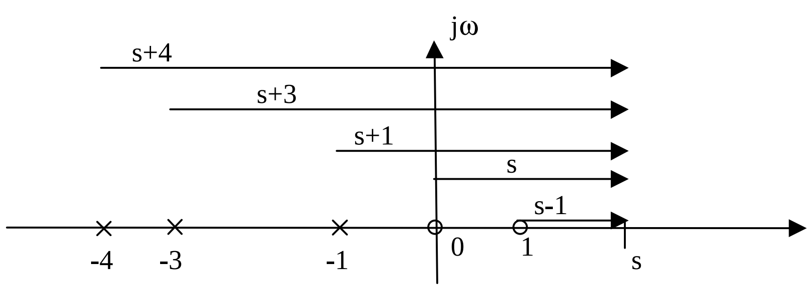
Consider the example

$$G(s) = \frac{s(s-1)}{(s+1)(s+3)(s+4)}.$$

The pole/zero plot is shown.



Let us examine the phase for values of *s* on the real axis *w=0*. For any value of *s* greater than *s=1*, one draws vectors for the pole/zero factors as shown.



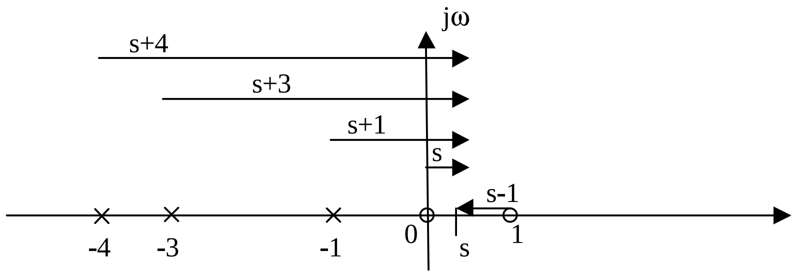
The vectors all lie along the real axis. They are offset simply so they can be visualized more easily.

The angle of *G(s)* is given by

$$\begin{aligned} \angle G(s) &= \angle(s-1) + \angle s - \angle(s+1) - \angle(s+3) - \angle(s+4) \\ &= 0+0+0+0+0 = 0^o \end{aligned}.$$

This means that *G(s)* is real and positive.

For any value of *s* between 0 and 1, one draws vectors for the pole/zero factors as shown.



The angle of *G(s)* is given by

$$\begin{aligned}\angle G(s) &= \angle(s-1) + \angle s - \angle(s+1) - \angle(s+3) - \angle(s+4) \\ &= 180^o + 0 + 0 + 0 + 0 = 180^o\end{aligned}.$$

This means that *G(s)* is real and negative.

For any value of *s* between -1 and 0, one has

$$\begin{aligned}\angle G(s) &= \angle(s-1) + \angle s - \angle(s+1) - \angle(s+3) - \angle(s+4) \\ &= 180^o + 180^o + 0 + 0 + 0 = 360^o = 0^o\end{aligned}.$$

Thus, again *G(s)* is real and positive.

In similar fashion, one may see that *G(s)* is real and negative on the real axis to the left of an odd number of poles plus zeros. This is an important property we shall use have occasion to use.