ELEC 3500
Control Systems
Lecture 10: Discrete-time Systems Z-transform: Inverse Z Transform, Discretization of Continuous-Time Controllers

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Z-Transform

Given a time series $y_k u_{-1}(k) = y_0, y_1, y_2, \cdots$, its Z-transform is given by

$$Y(z) = \sum_{k=0}^{\infty} y_k z^{-k} .$$

The discrete time variable is k. The one-sided transform is used (i.e. lower limit of zero) since the time series is multiplied by the discrete unit step $u_{-1}(k)$. The Z-transform variable z is a complex variable, like the Laplace transform variable $s = \sigma + j\omega$.

In connection with finding Z-transforms it is useful to recall the series identity

$$\sum_{k=0}^{N-1} a^k = \frac{1 - a^{N} \to 0}{1 - a} .$$

If $|a| < 1$, $|a|^N$, when N is large $\Rightarrow 0$

If $|a| < 1$, the series converges and one has also

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1 - a} .$$

Table of Z Transforms

Unit pulse 1,0,0,…		1
Unit step $u_{-1}(k)=1,1,1,1,\ldots$		$\dfrac{1}{1-z^{-1}}$
Discrete ramp k= 0,1,2,3,…		$\dfrac{z^{-1}}{(1-z^{-1})^2}$
Discrete exponential a^k		$\dfrac{1}{1-az^{-1}}$
Sampled Signals, T= sampling period		
$\sin\omega t$	$\sin\omega kT$	$\dfrac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
$\cos\omega t$	$\cos\omega kT$	$\dfrac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
$e^{-\alpha t}\sin\omega t$	$e^{-\alpha kT}\sin\omega kT$	$\dfrac{e^{-\alpha T}z^{-1}\sin\omega T}{1-2e^{-\alpha T}z^{-1}\cos\omega T+e^{-2\alpha T}z^{-2}}$
$e^{-\alpha t}\cos\omega t$	$e^{-\alpha kT}\cos\omega kT$	$\dfrac{1-e^{-\alpha T}z^{-1}\cos\omega T}{1-2e^{-\alpha T}z^{-1}\cos\omega T+e^{-2\alpha T}z^{-2}}$

From $y_k = a^k$ to $\frac{1}{1 - a z^{-1}}$

$$Y(z) = \sum_{k=0}^{\infty} \underset{\substack{\downarrow \\ y_k}}{a^k} z^{-k} = \sum_{k=0}^{\infty} (a z^{-1})^k = \frac{1}{1 - a z^{-1}} \quad //$$

$$= \frac{z}{z - a}$$

❑ Inverse Z-Transform by Partial Fraction Expansion

Example 1

$$Y(z) = \frac{z}{z^2 - 1.7z + 0.72}.$$

$$= \frac{z}{(z-0.9)(z-0.8)}$$

$$= \left(\frac{k_1}{z-0.9} + \frac{k_2}{z-0.8}\right) z$$

$$= Y_1(z) \cdot z$$

$$K_1 = \underline{(z-0.9)Y_1(z)}\Big|_{z=0.9} = (z-0.9)\frac{1}{\underline{z^2-1.7z+0.72}}\Bigg|_{z=0.9}$$

$$= \cancel{(z-0.9)}\frac{1}{\cancel{(z-0.9)}\underline{(z-0.8)}}\Bigg|_{z=0.9}$$

$$= \frac{1}{0.1} = 10$$

$$K_2 = (z-0.8)Y_1(z)\Big|_{z=0.8} = \frac{1}{z-0.9}\Bigg|_{z=0.8} = -10$$

$$Y(z) = \left(\frac{10}{z-0.9} + \frac{-10}{z-0.8} \right) z$$

$$= 10 \cdot \frac{z}{z-0.9} - 10 \frac{z}{z-0.8} \Rightarrow \frac{z}{z-a}$$

$$\frac{1}{1-a/z}$$

$$y_k = 10(0.9)^k - 10(0.8)^k, \quad k = 0, 1, 2, \ldots \infty$$

❑ Difference Equations

A linear time-invariant system may be described in discrete time in terms of its DISCRETE TRANSFER FUNCTION H(z). Then the input U(z) and output Y(z) are related by

Discrete-time polynomials

$$Y(z) = H(z)U(z) = \boxed{\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}} U(z)$$

degree is m

degree is n

So the relative degree of Y(z) is n-m

Dividing through by z^n one may write

dividing through by z^n yields

$$H(z) = \frac{b_0 z^{m-n} + b_1 z^{m-n-1} + \cdots + b_{m-1} z^{1-n} + b_m z^{-n}}{1 + a_1 z_1^{-1} + \cdots + a_{n-1} z^{1-n} + a_n z^{-n}}$$

Define system delay $d \triangleq n-m$, so

$$\frac{Y(z)}{U(z)} = H(z) = \frac{b_0 z^{-d} + b_1 z^{-d-1} + \cdots + b_{m-1} z^{1-d-m} + b_m z^{-d-m}}{1 + a_1 z^{-1} + \cdots + a_{n-1} z^{1-d-m} + a_n z^{-d-m}}$$

$$Y(z)\left[1 + a_1 z^{-1} + \cdots + a_{n-1} z^{1-d-m} + \underline{a_n z^{-d-m}}\right]$$

$$= \underline{U(z)\Big[b_0 z^{-d}} + b_1 z^{-d-1} + \cdots + \underline{b_m z^{-d-m}}\Big]$$

The Z-transform SHIFT PROPERTY says that multiplying the Z-transform by z^{-1} delays the time series by one. Compare this to the Laplace transform property which says that multiplying the transform by 1/s amounts to integrating the time function.

$$Y(z)\, z^{-1} \Rightarrow y_{k-1}$$

$$Y(z) \cdot z^{-n} \Rightarrow y_{k-n}$$

$$Y(z) \cdot z^{-m} \Rightarrow y_{k-m}$$

$$y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n-m} = b_0 u_{k-d} + b_1 u_{k-d-1} + \cdots + b_m u_{k-d-m}$$

Example 2

$$H(z) = \frac{z}{z^2 - 1.7z + 0.72}$$

seep 1: dividing through by z^2

$$H(z) = \frac{z^{-1}}{1 - 1.7z^{-1} + 0.72z^{-2}} \quad ① \quad = \frac{Y(z)}{U(z)}$$

seep 2: $H(z) = \frac{Y(z)}{U(z)}$ ②

From ① and ②, we obtain

$$z^{-1} U(z) = \left(1 - 1.7 z^{-1} + 0.72 z^{-2}\right) Y(z)$$

$$u_{k-1} = y_k - 1.7 y_{k-1} + 0.72 y_{k-2} \quad , \quad k \geq 2$$

or $$u_{k+1} = y_{k+2} - 1.7 y_{k+1} + 0.72 y_k \quad , \quad k \geq 0$$

❑ Discretization of Continuous-Time Controllers

The relation between the Laplace transform variable s and the Z-transform variable z is $z=e^{sT}$, with T the sampling period. However, using this to transform K(s) to K(z) will give non-polynomial transfer functions. Note that

$$\boxed{e^{sT} \approx \frac{1+ sT/2}{1- sT/2}}$$

Therefore define the BLT by

$$z = \frac{1+ sT/2}{1- sT/2},$$

and its inverse

$$s = \frac{2}{T}\frac{z-1}{z+1}.$$

To convert a continuous transfer function K(s) to a discrete transfer function using sample period T, then, one simply replaces all occurrences of s by $\frac{2}{T}\frac{z-1}{z+1}$. That is

$$K(z) = K(s)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}}.$$

Example 3

A continuous-time low pass filter is given by

$$K(s) = \frac{\alpha}{s+\alpha}.$$

$\Rightarrow$ pole $-\alpha$, zero 0

What is $K(z)$?

$$K(z) = \left. \frac{\alpha}{s+\alpha} \right|_{s = \frac{2}{T} \cdot \frac{z-1}{z+1}}$$

$$= \frac{\alpha}{\frac{2}{T} \cdot \frac{z-1}{z+1} + \alpha} = \frac{(z+1)T\alpha}{2(z-1) + T(z+1)}$$

$$= \frac{(z+1)T\alpha}{z(2+T)+T-2}$$

pole : $z(2+T)+T-2=0$

$$z = \frac{2-T}{2+T}$$

zero : -1