ELEC 3500
Control Systems
Lecture 9: State-Variable Feedback: Reachability and Pole Placement by Ackermann's Formula

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❑ State-Variable Feedback (SVFB)

$$\dot{x} = Ax + \underline{B}u$$

$$y = Cx + Du$$

$x \in R^n$, $u \in R^m$, $m < n$

$y \in R^p$, $p < n$,

$$u = -kx + v$$

v is another external input

$$\dot{x} = Ax + Bu$$

closed-loop dynamics

$$= \underbrace{(A - Bk)}_{A_c}x + Bv$$

$$y = Cx + Du$$

$$= \underbrace{(C - Dk)}_{C_c}x + Dv$$

$$sX(s) = A_c X(s) + Bv$$

$$X(s) = (sI - A_c)^{-1}Bv$$

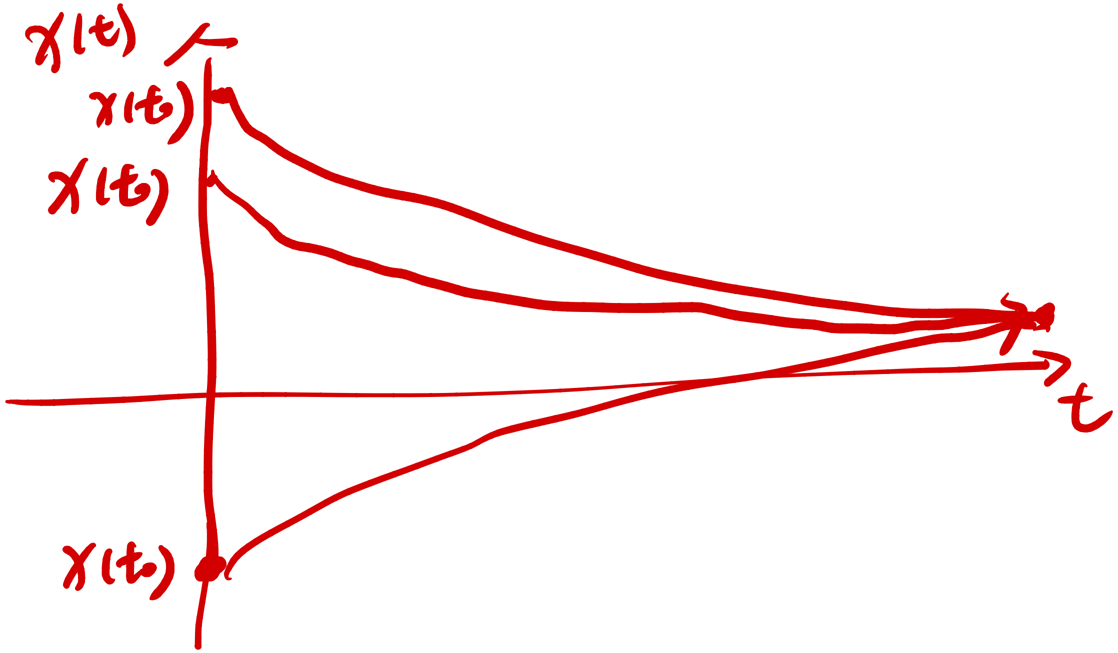
$$\Delta_c(s) = (sI - A_c)^{-1}$$

❑ State-Variable Feedback (SVFB)

$$H_c(s) = C_c (sI - A_c)^{-1} B + D$$

❑ Reachability

A system is said to be REACHABLE (sometimes called CONTROLLABLE) if the control input u(t) can be selected to drive the state x(t) from any given initial condition to any desired final value.



Reachable matrix

$U = [\, B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B \,]$, n is the state dimension

A system is reachable if only if $\operatorname{rank}(U) = n$

For example, $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}_{2\times 2}, \quad B = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$U = \begin{bmatrix} B & \underline{AB} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

$\operatorname{rank}(U) = 2$

reachable

❑ SVFB Pole Placement and Ackermann's Formula

POLE-PLACEMENT problem

Select K such that $\Delta_c(s) = |sI - A_c| = |sI - (A - BK)|$ has prescribed poles.

ACKERMANN'S FORMULA

$$K = e_n U^{-1} \Delta_D(A) \, ,$$

n is state dimension

where $e_n = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$ is the last row of the $n \times n$ identity matrix, and $\Delta_D(s)$ is the DESIRED characteristic polynomial. The desired characteristic polynomial evaluated at the plant matrix A is $\Delta_D(A)$, which is a MATRIX POLYNOMIAL.

❑ Example 1

The angle subsystem of the inverted pendulum is given for some specific values of parameters by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u$$

Design a SVFB K that will give a closed-loop POV of 4% with a settling time of $t_s = 1$ sec.

$$u = -Kx$$

By ~~Acker~~ Ackermann's formula,

$$K = e_2 U^{-1} \Delta_D(A)$$

$$e_2 = [\,0 \quad 1\,]$$

$$U^{-1} = [B \quad AB]^{-1} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{-1} = \frac{1}{-4}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

To find

$\Delta_D(A)$, we need to find $\Delta_D(s)$

$$\Delta_D(s) = s^2 + 2\xi\omega_n + \omega_n^2$$

$$POV = 4\% \longrightarrow \xi = 0.7$$

settling time $1s \rightarrow t_s = 1s = 5\tau = 5 \cdot \frac{1}{\alpha} \rightarrow \alpha = 5$

$$\omega_n = \frac{\alpha}{\xi} = \frac{5}{0.7} = 7.07$$

$$\Delta_D(s) = \underline{\underline{s^2 + 10s + 50}}$$

$$\Delta_D(A) = \Delta_D(s)\Big|_{s=A}$$

$$= A^2 + 10 \cdot A + 50 \cdot I_{2\times 2}$$

$$= \begin{bmatrix} 59 & 10 \\ 90 & 59 \end{bmatrix}$$

$$K = e_2 U^{-1} \Delta_D(A) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix} \begin{bmatrix} 59 & 10 \\ 90 & 59 \end{bmatrix}$$

$$= \begin{bmatrix} -29.5 & -5 \end{bmatrix}$$

To double-check if the k results in the above POV = 4%, ts = 1s

$$\Delta(s) = \left| sI - (A - BK) \right|$$

$$= s^2 + 10s + 50 \quad //$$