ELEC 3500
Control Systems
Lecture 8: Feedback Control Systems: Pole Placement, Disturbance Rejection, Steady-State Error

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❑ POLE PLACEMENT USING FEEDBACK

Move the closed-loop poles and improve the system performance:
fast transient response, percent overshoot, oscillation frequency

❑ DISTURBANCE REJECTION

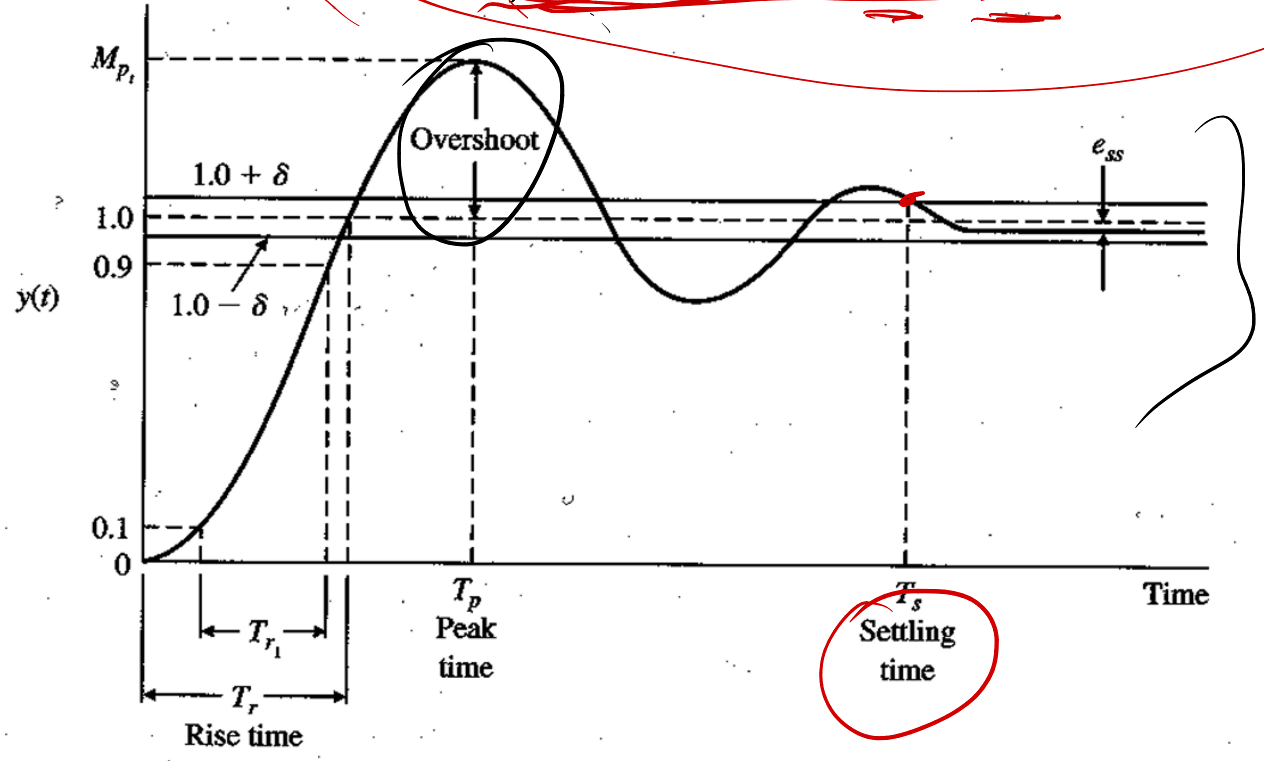
Reduce the effects of disturbances on the output: wind gusts in aircraft control systems

❑ Steady-State Error

Select a suitable compensator to obtain good steady-state tracking by studying the steady-state error

$$H(s) = \frac{k_0\omega_n^2}{s^2 + 2\alpha s + \omega_n^2} = \frac{k_0\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \equiv \frac{k_0\omega_n^2}{\Delta(s)}$$

$$\beta = \sqrt{\omega_n^2 - \alpha^2} = \omega_n\sqrt{1-\zeta^2} \qquad \zeta = \frac{\alpha}{\omega_n} \qquad T = \frac{2\pi}{\beta}$$



Sketch of the step response of a second order system with complex poles

Percent overshoot (POV)

$$POV = \frac{y_{max} - y_{ss}}{y_{ss}} \times 100\%$$

$$y_{ss} = \lim_{t\to\infty} y(t) = k_0$$

$$\zeta = \left[\frac{\ln^2\left(POV/100\right)}{\ln^2\left(POV/100\right) + \pi^2}\right]^{1/2}$$

$$POV = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$t_r = 2.2\tau \qquad t_r = \frac{2.16\zeta + 0.6}{\omega_n}$$

Rise time t_r: the time required for the step response to rise from 0.1 to 0.9 of its steady-state value.

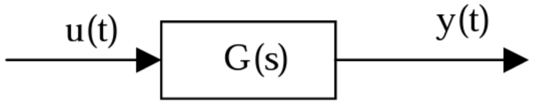
Settling time t_s: the time required for the signal to effectively reach its steady-state value.

$$t_s = 5\tau \qquad \tau = 1/\alpha$$

POLE PLACEMENT USING FEEDBACK

Example 1- An open-loop plant has transfer function

$$G(s) = \frac{1}{s^2 - s + 1}$$



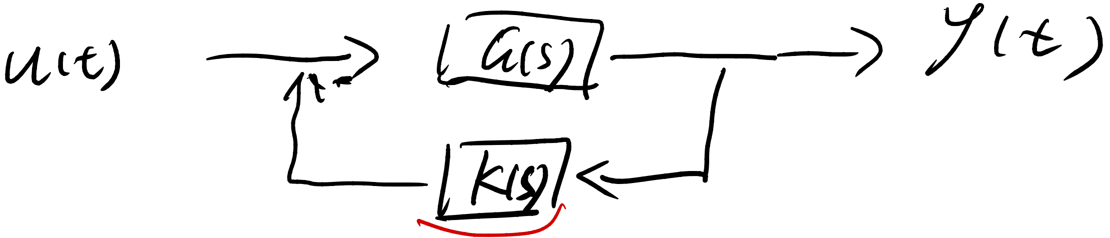
$2\alpha = -1 \quad \alpha = -\frac{1}{2}$

We want to select the gains to obtain the performance specifications: closed-loop POV of 4% and settling time of 2 sec.

$2 = 5\tau \quad \tau = 0.4$

$\alpha = \frac{1}{0.4}$

step1: design feedback



Assume $K(s) = K_D s + K_P$

parameters

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K(s)\cdot G(s)}{1+K(s)G(s)}$$

$$H(s) = \frac{K(s)\,G(s)}{1+(K_D s + K_P)\cdot \dfrac{1}{s^2-s+1}}$$

$$= \frac{(s^2-s+1)\,K(s)\,G(s)}{s^2-s+1+K_D s+K_P}$$

$$= \frac{\quad}{s^2+(K_D-1)s+K_P+1}$$

step 2 : find K_P, K_D using performance specifications

$POV = 4\%$ $\quad \xi = \left[\frac{\ln^2(POV/100)}{\ln^2(POV/100)+\pi^2}\right]^2, \quad \xi = \frac{1}{\sqrt{2}}$

Settling time: $t_s = 2$ sec $\quad t_s = 5\tau, \ \tau = \frac{1}{\alpha} \Rightarrow \alpha = \frac{5}{2} = 2.5$

$$\omega_n = \frac{\alpha}{\xi} = \frac{5}{2}\cdot\sqrt{2} = \frac{5}{\sqrt{2}}$$

$$\Delta^{des}(s) = s^2 + 2\alpha s + \omega_n^2 = s^2 + 5s + \frac{25}{2}$$

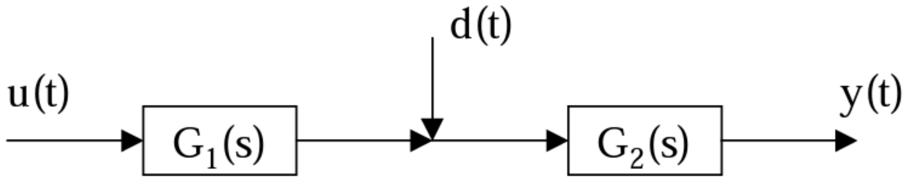
$\downarrow$

desired $\Delta(s)$

So $\quad K_D - 1 = 5 \quad \Rightarrow \quad K_D = 6$

$\quad K_P + 1 = \frac{25}{2} \quad \Rightarrow \quad K_P = \frac{23}{2}$

❑ DISTURBANCE REJECTION

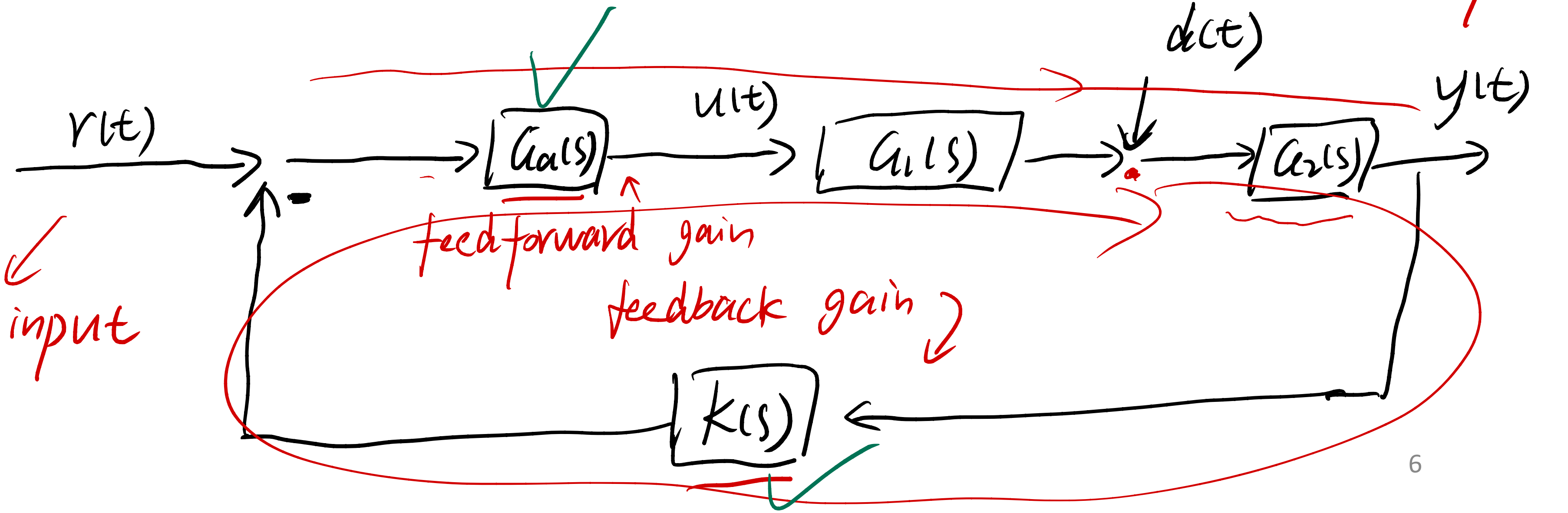


$$Y(s) = G_1 G_2 U + G_2 D$$

To reduce the effect of the disturbance d(t) on the output y(t) and also produce a desirable transfer from the control to y(t), we

disturbance input

Design



$$Y(s) = H_D(s)\,D(s) + H_r(s)\cdot R(s)$$

transfer function from $d(t) \to y(t)$

transfer function from $r(t) \to y(t)$

$$H_D(s) = \frac{G_2}{1 + K G_a G_1 G_2}$$

$$H_r(s) = \frac{G_a G_1 G_2}{1 + K G_a G_1 G_2}$$

Example 2- Compensator Design Example

a. Open-Loop System.

The plant is an aircraft with lightly damped modes given by

$$G_2(s) = \frac{s}{(s+0.1)^2+1}$$

and the actuator is a motor that is modeled as

$$G_1(s) = \frac{1}{s+20}$$

b. Performance Specifications.

Design a 2-DOF regulator so that, in closed loop:

$t_s = 0.5\,s$

1. The disturbance dies away in approximately 0.5 second.
2. The transfer function from r(t) to y(t) has a damping ratio of $\zeta = 1/\sqrt{2}$ and a natural frequency of $\omega_n = \sqrt{2}$ sec. These figures give a response that many pilots like, in terms of a good compromise between overshoot and speed of response.

1. The disturbance dies away in approximately 0.5 second.

Settling time 0.5 s

$t_s = 0.5 = 5\tau, \quad \tau = 0.1$

$\alpha = \frac{1}{\tau} = 10, \quad H_D^{des} = \frac{1}{0.1s+1} = \frac{10}{s+\underset{\alpha}{\underline{\underline{10}}}} \triangleq$

2. The transfer function from r(t) to y(t) has a damping ratio of $\zeta = 1/\sqrt{2}$ and a natural frequency of $\omega_n = \sqrt{2}$ sec. These figures give a response that many pilots like, in terms of a good compromise between overshoot and speed of response.

$\zeta = \frac{1}{\sqrt{2}}, \quad \omega_n = \sqrt{2}$

$$H_r^{des} = \frac{s}{s^2 + 2 \cdot \frac{1}{\sqrt{2}} \sqrt{2} \cdot s + (\sqrt{2})^2} = \frac{s}{s^2 + 2s + 2}$$

So, Let

$$H_D(s) = H_D^{des}$$

$$H_r(s) = H_r^{des}$$

which are

$$\frac{G_2}{1 + K G_a G_1 G_2} = \frac{1}{s+10} \quad (1)$$

$$\frac{G_a G_1 G_2}{1 + K G_a G_1 G_2} = \frac{s}{s^2+2s+2} \quad (2)$$

$$\frac{②}{①} \Rightarrow \quad G_a G_1 = \frac{s(s+10)}{s^2+2s+2}$$

$$G_a = \frac{s(s+10)(s+20)}{s^2+2s+2}$$

From ①, we have

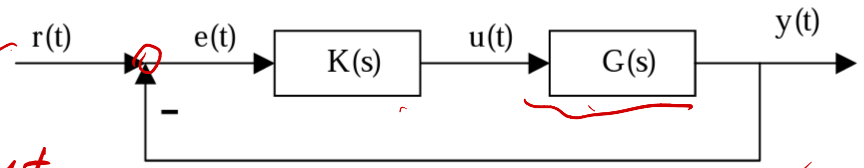
$$G_2(s+10) = 1 + G_a K G_1 G_2$$

$$K = \frac{(s+10)G_2 - 1}{G_1 G_2 G_a}$$

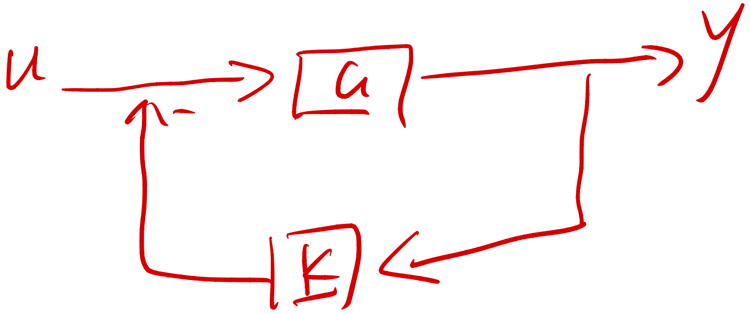
❑ Steady-State Error

$$\frac{G}{1+KG}$$

Feedback can improve the tracking capabilities of a plant by making the steady-state error smaller.



ref input



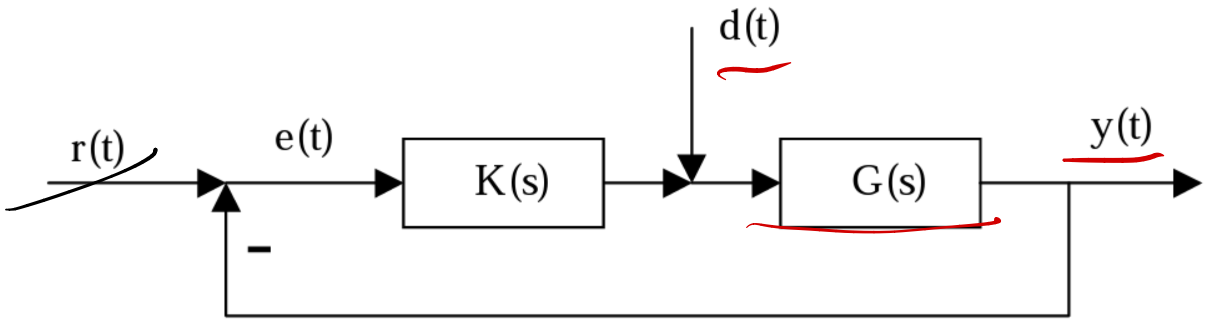
$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG}{1+KG}$$

Define tracking error

$$e(t) = r(t) - y(t)$$

Then, $E(s) = R(s) - Y(s)$

Example 3- Steady-State Error Analysis



$$\frac{G(s)}{1+KG}$$

In this closed-loop system, the plant is

$$G(s) = \frac{1}{s+1}$$

and the controller is the integral compensator

$$K(s) = \frac{k_I}{s}$$

with k_I the integral gain.

$$Y(s) = H_D(s) \cdot D(s) + H_r(s) \cdot R(s)$$

a. Find the Steady-State Error in Response to a Unit Step Disturbance *d(t)*

$D(s) = \frac{1}{s}$

Assume $R(s) = 0$

$$E(s) = R(s) - Y(s) = -Y(s) = -H_D(s) \cdot D(s)$$

$$= -\frac{G}{1+KG} \cdot \frac{1}{s}$$

Because $G(s) = \frac{1}{s+1}$,

$K(s) = \frac{k_I}{s}$

Then, $E(s) = \frac{-1}{s^2+s+k_I}$

- Initial value theorem

$$\lim_{s \to \infty} sX(s) = x(0)$$

$x(t)$ time function

$X(s)$ its Laplace transform

- Final value theorem

$$\lim_{s \to 0} sX(s) = \lim_{t \to \infty} x(t)$$

So, $e(\infty) = \lim_{s \to 0} sE(s)$

$$= \lim_{s \to 0} \frac{-s}{s^2 + s + k_I}$$

$$= 0$$

b. Find the Steady-State Error in Response to a Unit Step Command *r(t)*

Regard $D(s) = 0$ $R(s) = \frac{1}{s}$

$$E(s) = R(s) - Y(s)$$

$$= R(s) - H_r R(s) - \cancel{H_D D(s)}$$

$$= (1 - H_r) R(s)$$

$$= \left(1 - \frac{K(s)G(s)}{1 + K(s)G(s)}\right) R(s)$$

$$= \frac{1}{1 + K(s)G(s)} \cdot R(s) = \frac{s+1}{s^2 + s + k_I}$$

$$E(t), \quad E(\infty) = \lim_{s \to 0} s \cdot E(s)$$

$$= \lim_{s \to 0} \frac{s(s+1)}{s^2 + s + K_I}$$

$$= 0$$

c. Find the POV For a Unit Step Command.

$r(t)$

$R(s) = \frac{1}{s}$

$R(s) = 0$

$$Y(s) = H_r(s) \cdot R(s)$$

$$= \frac{k_I}{s^2 + s + k_I} \frac{1}{s}$$

$$= \frac{k_I / s}{s^2 + 2\xi \omega_n s + \omega_n^2}$$

$$POV = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\begin{cases} \omega_n^2 = k_I \\ 2\xi\omega_n = 1 \end{cases}$$

$$\omega_n = \sqrt{k_I}$$

$$\xi = \frac{1}{2\sqrt{k_I}}$$

d. Find the Output $y(t)$ if $r(t) = e^{-t}u_{-1}(t)$, $d(t) = e^{-2t}u_{-1}(t)$

$$Y(s) = H_r(s)R(s) + H_D(s)D(s)$$

$$= H_r(s)\cdot\frac{1}{s+1} + H_D(s)\cdot\frac{1}{s+2}$$

$$H_r(s) = \frac{G}{1+KG} = \frac{s}{s^2+s+k_I}$$

$$H_D(s) = \frac{GK}{1+KG} = \frac{k_I}{s^2+s+k_I}$$

Then, do PFE to find $y(t)$