REALIZATION AND CANONICAL FORMS

A linear time-invariant (LTI) system can be represented in many ways, including:

- differential equation (ODE)
- state variable (SV) form
- transfer function
- impulse response
- block diagram (BD) or flow graph

Each description can be converted to the others. In the lecture on Mason's Formula we saw how to represent systems in terms of block diagrams, and how to determine the transfer function of a block diagram system using Mason's Formula.

In this lecture we shall see how to make a block diagram from a given transfer problem. This is the inverse problem from the one Mason's Formula solves. Then, we shall see how to associate a state-space equation with any block diagram. The relation between SV systems and BD is very transparent since the output of each integrator is a state.

The problem of finding a SV or BD representation given a prescribed transfer function is called the *realization problem*.

BD REALIZATION OF TRANSFER FUNCTIONS IN SERIES FORMS

A transfer function can be realized as a BD in series form or parallel form. Here we introduce two series forms that are very convenient for solving the BD realization problem for single-input/single-output (SISO) systems.

Consider the illustrative third-order transfer function

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}. \qquad \textbf{(1)}$$

This is a *rational* function (e.g. a ratio of two polynomials in *s*). For realization, it is important to ensure that the transfer function is *monic*, that is, the highest order term in the denominator has a coefficient of 1. If not, divide through by this coefficient to put the transfer function in monic form.

The transfer function must also have relative degree of 1 or more. If the relative degree is zero (e.g. same power of *s* in the numerator as the denominator), then divide the denominator into the numerator in one step of long division to write *H(s)* as a constant term plus a term whose relative degree is at least one. The constant term is a direct feedthrough term, and the procedures below may be carried out to realize the remainder term.

A transfer function is said to be *proper* if its relative degree is greater than or equal to zero, and *strictly proper* if the relative degree is greater than or equal to one.

We use a third-order system to illustrate the approach, which works for any *n*-th order rational, monic, strictly proper transfer function.

To find a BD realization of *H(s)*, divide by the highest power of *s* to obtain

$$H(s) = \frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}} = \frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 - (-a_2 s^{-1} - a_1 s^{-2} - a_0 s^{-3})}. \qquad \textbf{(2)}$$

Now think of Mason's Formula. To draw a BD we can use three feedforward paths and three loops if we select the correct transmissions and loop structure.

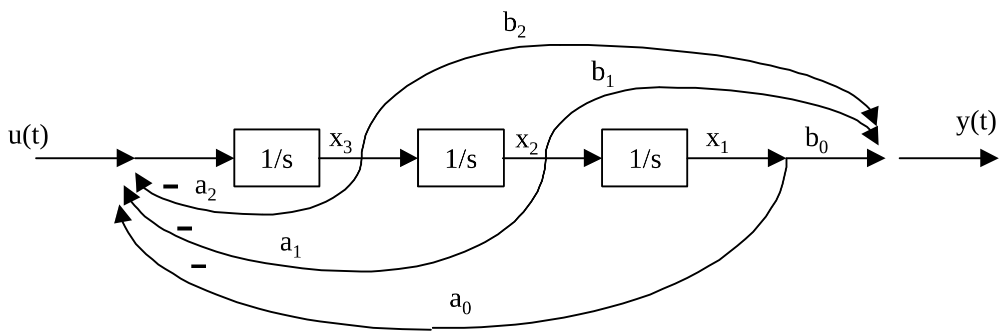
We give two series forms that have a convenient structure for realizing SISO systems. Note particularly that Mason's Formula is very easy to use if there are no disjoint loops, and all loops touch all feedforward paths. Then, the determinant *D(s)* is simply 1 minus the sum of the loop gains, and all cofactors are equal to one.

Reachable Canonical Form (RCF)

The rule used for RCF is

feedback to the left
feedforward to the right.

This means that all feedback loops should join at a summer on the left, and all feedforward paths should join at a summer on the right. A BD satisfying this condition is drawn below.



Reachable Canonical Form

Note that all loops and all feedforward paths have the left-hand integrator in common, so all cofactors are equal to 1 and the determinant has no higher-order terms. Applying Mason's Formula to this BD gives the transfer function (2). Determine the loop gains and path gains and make sure you believe this.

Each integrator output is labeled as a state. The rule used in this course for labeling states will be:

Label the states from right to left, from top to bottom.

We will see some examples of this to clarify it.

With the states labeled as shown, one may write down directly the state equations

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u = Ax + Bu$$

(3)

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} x \qquad = Cx$$

As an exercise, one may find the transfer function

$$H(s) = C(sI - A)^{-1}B + D$$

and verify that it is the same as the one we started with.

This development gives a very easy way to realize SISO transfer functions in SV form. One notes that it is easy to write down (3) directly from (1) without having to draw the BD. In fact, simply take the denominator of *H(s)*, turn the coefficients backwards, make them negative, and place them into the bottom row of the *A* matrix. Take the coefficients of the numerator, turn them backwards, and place them into the *C* matrix.

The *A* matrix in (3) is known as a *bottom companion matrix* for the characteristic polynomial

$$\Delta(s) = s^3 + a_2 s^2 + a_1 s + a_0 .$$

The superdiagonal 1's in *A* and the lower 1 in *B* mean simply that the three integrators are connected in series.

Example 1. Realize Transfer Function as RCF SV System

Let there be prescribed

$$H(s) = \frac{s^2 + 2s - 1}{s^3 + 2s^2 + 3s + 4} .$$

The SV equations are directly written down as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} x \qquad = Cx$$

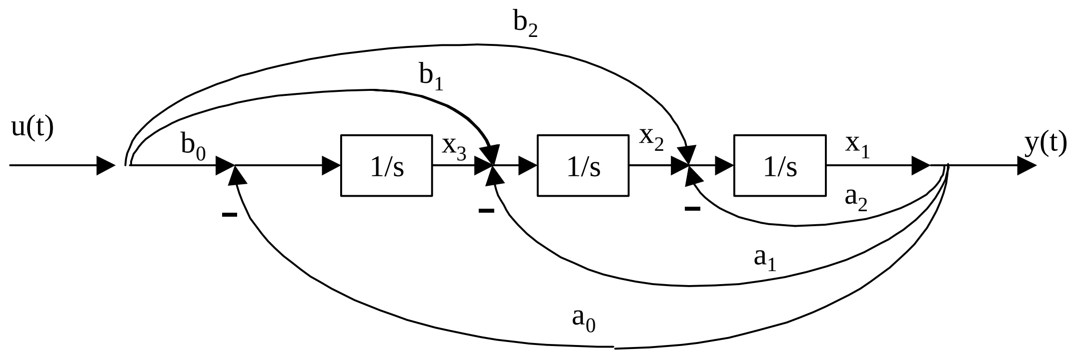
Now one may analyze the system including simulation, finding output given an input and ICs, etc.

Observable Canonical Form (OCF)

The rule used for OCF is

feedback from the right
feedforward from the left

A BD satisfying this condition is drawn below.



Observable Canonical Form

Note that all loops and all feedforward paths have the right-hand integrator in common, so all cofactors are equal to 1 and the determinant has no higher-order terms. Applying Mason's Formula to this BD gives the transfer function (2). Determine the loop gains and path gains and make sure you believe this.

With the states labeled from right to left as shown, one may write down directly the state equations

$$\dot{x} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \qquad = Cx \tag{4}$$

As an exercise, one may find the transfer function

$$H(s) = C(sI - A)^{-1} B + D$$

and verify that it is the same as the one we started with.

This development gives a very easy way to realize SISO transfer functions in SV form. One notes that it is easy to write down (4) directly from (1) without having to draw the BD. In fact, simply take the denominator of *H(s)*, stack the coefficients on end, make them negative, and place them into the first column of the *A* matrix. Take the coefficients of the numerator, stack them on end, and place them into the *B* matrix.

Note that this OCF state-space form is not the same as RCF, though both have the same transfer function. In fact, RCF and OCF are related by a *state-space transformation*, which we shall not discuss in this course (it is discussed in EE 5307, Linear Systems).

The A matrix in (4) is known as a *left companion matrix* for the characteristic polynomial

$$\Delta(s) = s^3 + a_2 s^2 + a_1 s + a_0 .$$

The superdiagonal 1's in A and the left-hand 1 in C mean simply that the three integrators are connected in series.

Example 2. Realize Transfer Function as OCF SV System

Let there be prescribed

$$H(s) = \frac{s^2 + 2s - 1}{s^3 + 2s^2 + 3s + 4} .$$

The SV equations are directly written down as

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \qquad = Cx$$

Now one may analyze the system including simulation, finding output given an input and ICs, etc.

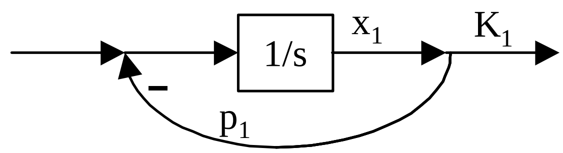
BD REALIZATION OF TRANSFER FUNCTIONS IN PARALLEL FORM

To realize a system in parallel form, one performs a PFE on the transfer function to obtain

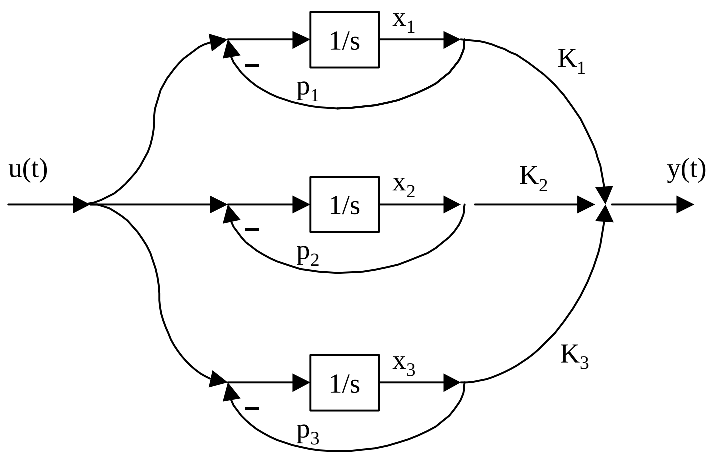
$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} + \frac{K_3}{s + p_3} \qquad \textbf{(5)}$$

where the poles are at $s = -p_1, -p_2, -p_3$ and the residues are K_1, K_2, K_3.

Now note that a single term of this form can be realized using the simple BD shown. Use Mason's Formula to determine the transfer function to make sure you believe this.



The complete transfer function with three parallel paths can be realized as shown.



This realization is known as parallel form. If there are repeated poles, then the transfer function has higher-order poles in the PFE. In this event, some parallel paths will contain multiple integrators. We do not need to know the details for this course.

A system which has a PFE with no higher-order poles is called *simple*.

The parallel form is known as *Jordan Normal Form* in mathematics. The case of higher-order pole factors, corresponding to multiple integrators in some paths, corresponds to what is known as *eigenvector chains* in those paths. The details, though fascinating, are not needed in this course.

With the states labeled from top to bottom as shown, one may write down directly the state equations

$$\dot{x} = \begin{bmatrix} -p_1 & 0 & 0 \\ 0 & -p_2 & 0 \\ 0 & 0 & -p_3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} x \qquad = Cx \tag{6}$$

As an exercise, one may find the transfer function

$$H(s) = C(sI - A)^{-1}B + D$$

and verify that it is the same as the one we started with.

Note that for a scalar system (e.g. $n=1$) one may write

$$H(s) = \frac{cb}{s-a} + d\ .$$

This shows that the residues can be placed on the input paths in the figure above. In fact, as long as $cb = K$ in each path, one can split the residues between input and output paths.

This development gives a very easy way to realize SISO transfer functions in SV form. One notes that it is easy to write down (6) directly from (5) without having to draw the BD.

Note that this parallel state-space form is not the same as RCF or OCF, though all three have the same transfer function. In fact, RCF, OCF, and the Jordan form are related by *state-space transformations*, which we shall not discuss in this course.

The A matrix in (6) is known as a *parallel form matrix* for the characteristic polynomial

$$\Delta(s) = s^3 + a_2 s^2 + a_1 s + a_0 = (s+p_1)(s+p_2)(s+p_3)\ .$$

If the A matrix is diagonal with the poles appearing on the diagonal it is called *simple*. If the system is not simple, then there will be some off diagonal 1's in A corresponding to Jordan eigenvector chains of length greater than one. You will hear more about this in another course (EE 5307).

STABILITY OF LINEAR SYSTEMS

We discuss the stability of input/output systems and of state-space systems. The Routh Test is given for finding the number of right-half plane roots of a prescribed polynomial.

Stability of Input/Output Systems

Input/output systems may be described by a transfer function H(s) or equivalently by an impulse response h(t).

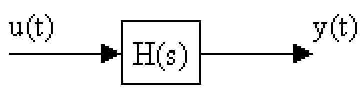


Figure 1

The system is said to be:

Asymptotically Stable (AS) if u(t)= 0 for all time t implies that y(t) goes to zero with time.

Marginally Stable (MS) if u(t)= 0 for all time t implies that y(t) is bounded for all time.

In terms of time signals, one could say that a decaying exponential is AS, a sine wave is MS, the unit step is MS, and an increasing exponential is unstable.

A system is AS if and only if (iff) the impulse response goes to zero with time, and MS iff the impulse response is bounded. The natural modes that occur in h(t) depend on the locations of the poles, defined as the roots of the denominator of H(s). Thus, the system is AS iff all poles are in the open left-half plane (OLHP) (e.g. strictly in the LHP with none on the $j\omega$-axis). The system is MS iff all poles are in the left-half plane (e.g.

they may be in the OLHP or on the jω-axis), with any poles occurring on the jω-axis non-repeated.

Examples:

1. The unit step has transform 1/s, which has one pole at s=0 and is MS. The unit ramp has transform $1/s^2$, which has two poles at s=0 and is unstable.

2. The transfer function $H(s) = \dfrac{b}{(s+a)^2 + b^2}$ has poles at $s = -a \pm jb$ and is AS. Its impulse response is $e^{-at} \sin bt$, an exponentially decaying function.

3. The transfer function $H(s) = \dfrac{s}{s^2 + b^2}$ has poles on the jω-axis at $s = \pm jb$ and is MS. Its impulse response is $h(t) = \cos bt$, a bounded function.

4. The transfer function $H(s) = \dfrac{s}{(s^2 + b^2)^2}$ has repeated poles at $s = \pm jb$ and so is unstable. Its impulse response is $h(t) = t \cos bt$, an unbounded function.

Routh Test

Given a polynomial p(s), the number of poles in the right-half plane may be determined *without finding the roots* by using the Routh test.

Given $p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$, the Routh Array is given by

s^n	a_0	a_2	a_4	a_6 …
s^{n-1}	a_1	a_3	a_5	a_7 …
s^{n-2}	b_1	b_2	b_3	…
:	:	:	:	
s^0				

The third row and below are each computed from the two rows immediately preceding it by using the matrix determinantal equations, e.g. for the third row,

$$b_1 = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}, \quad b_3 = \frac{-\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1}.$$

Element a_1 is called the *pivot element* for row two. To simplify computations by avoiding fractions, one can at any point multiply any row by a positive constant before proceeding to the next row.

Routh Theorem. The number of roots of p(s) in the right-half plane equals the number of sign changes in column one.

To examine the input/output stability of a system, one applies the Routh test to the characteristic polynomial, the denominator of the transfer function. The Routh test was extremely useful for analyzing system stability in the days before calculators and computers. Finding the roots of high-order polynomials by hand is very difficult.

Example 1

Let a system have characteristic polynomial $p(s) = s^4 + 2s^3 + 6s^2 + 10s + 7$. How many roots are in the right-half plane?

The Routh array is manufactured as

s^4	1	6	7
s^3	2	10	
s^2	1	7	
s^1	-4		
s^0	7		

There are two sign changes in column one, so the polynomial has two unstable roots.

Example 2

Let $p(s) = s^2 - \frac{7}{4}s - \frac{1}{2}$. How many roots are in the right-half plane?

The Routh array is manufactured as

s^2	1	$-\frac{1}{2}$	
s^1	$-\frac{7}{4}$		
s^0	$-\frac{1}{2}$		

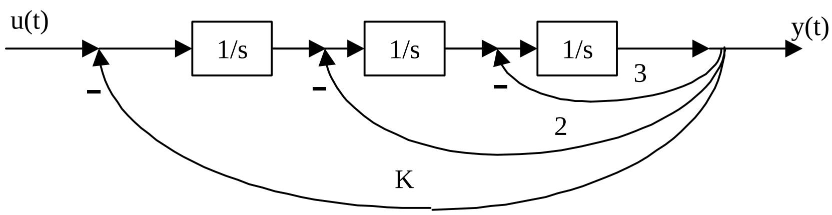
There is one sign change in column one, so the polynomial has one unstable root.

Note that, for stability, a second-order polynomial must have all signs the same.

Example 3- Stability as a Function of a Parameter

A very important application of the Routh test is determining the stability of a closed-loop system as a function of a variable parameter, which could be the feedback gain.

In the figure, determine for what values of feedback gain K the system is stable.



To solve this problem one simply determines the characteristic polynomial and then constructs the Routh Array.

Using Mason's Formula, one finds that the determinant is

$$\Delta(s) = 1 - \left(-\frac{3}{s} - \frac{2}{s^2} - \frac{K}{s^3} \right) = \frac{s^3 + 3s^2 + 2s + K}{s^3}.$$

The characteristic polynomial (denominator of the transfer function) is the top portion of this, so that

$$p(s) = s^3 + 3s^2 + 2s + K .$$

The Routh Array is constructed to be

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

This has no sign changes in column one if

$$\begin{aligned} 6 - K &> 0 \\ K &> 0 \end{aligned}.$$

Thus, for stability one requires

$0 < K < 6$.

Note that too much gain in this problem will destabilize the system, evidently making the poles unstable.

Routh Test Problem 1: Two Successive Rows Proportional

The Routh test can have two problems:

- Successive rows proportional
- Pivot element equal to zero (e.g. a zero in column one)

If two successive rows are proportional then, using the determinants, one will get the next row of the Routh Array equal to zero. In this case, one must take the second, linearly dependent, row out of the Routh Table, differentiate it, and put it back to obtain the row for the next lower power of *s*.

Example 4- Two Successive Rows Proportional

Let there be prescribed $p(s) = s^7 + 4s^6 + 5s^5 + 5s^4 + 6s^3 + 9s^2 + 8s + 2$. One computes the first part of the Routh Array as:

s^7	1	5	6	8	
s^6	4	5	9	2	
s^5	15	15	30		(This row was multiplied by 4 for simplicity)
s^4	1	1	2		
s^3	0	0	0		

Now all progress stops as one has obtained a zero row.

To remedy the problem and finish the Routh Array, take out the polynomial corresponding to s^4, which is

$$p_4(s) = s^4 + s^2 + 2.$$

Note the way one writes every other power of *s* to fabricate the polynomial. Now, differentiate this to obtain

$$p_3(s) = 4s^3 + 2s.$$

This is used as the row for s^3 in the array, since the original s^3 row came out as zero.

Proceeding to compute the rest of the Routh Array, one obtains:

s^7	1	5	6	8	
s^6	4	5	9	2	
s^5	15	15	30		(This row was multiplied by 4 for simplicity)
s^4	1	1	2		Take row out and differentiate
s^3	4	2			Put s^3 row back
s^2	$\frac{1}{2}$	2			
s^1	-14				
s^0	2	0	0		

There are two signs in column one, corresponding to two unstable roots.

This problem of two successive rows proportional occurs when the second row corresponds to an exact divisor of *p(s)*. In this example, the polynomial $p_4(s) = s^4 + s^2 + 2$ exactly divides *p(s)*. In fact, all the sign changes in the first column below the s^4 row refer to the roots of $p_4(s)$. Thus, the polynomial $s^4 + s^2 + 2$ has two roots in the right-half plane.

Routh Test Problem 2: Zero in Column One

If the pivot element of a row is equal to zero, one cannot divide by it to find the next row. In this event, one places a small positive nonzero number as the pivot element, represented by a Greek symbol such as e, and then proceeds. One can have several of the Greek letters in the same Routh Array if the problem occurs more than once in the same array.

Example 5- Zero Pivot Element

Let there be prescribed $p(s) = s^6 + 3s^5 + 2s^4 + 6s^3 + 3s^2 + 6s + 3$. One computes the first part of the Routh Array as:

s^6	1	2	3	3
s^5	3	6	6	
s^4	0	1	3	

One now has a zero in column one (but not an entire zero row, which occurs when two rows are proportional). Place e in column one, representing a small positive constant, and proceed to compute the Routh Array. In doing this, one may neglect terms in e^2 and higher, as they are very small. One also neglects terms in e when added to finite numbers. Finally, one makes use of the ability to multiply an entire row by a positive constant to simplify the computations.

s^6	1	2	3	3	
s^5	3	6	6		
s^4	e	1	3		
s^3	$6e-3\approx -3$	$6e-9\approx -9$			This row was multiplied by e
s^2	1	3			
s^1	x				Zero pivot: Place a small positive const. x in col. one
s^0	3				This row was multiplied by x

Note that in the s^1 row one again encounters a zero pivot element, so it is necessary to put another small positive variable into the table; here x was used.

Since the variables are assumed positive, there are two sign changes in the first column and this polynomial has two unstable roots.

Stability of State Variable Systems

The linear time invariant (LTI) state-space form is

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) \\ y &= Cx + Du\end{aligned}.$$

The transfer function is given by

$$H(s) = C(sI - A)^{-1}B + D.$$

The denominator of this is the characteristic polynomial

$$\Delta(s) = |sI - A|.$$

The system poles are the roots of the characteristic equation

$$\Delta(s) = |sI - A| = 0.$$

A state-space formulation allows one to get more information about the system than the input/output formulation, which is described only by a transfer function. Specifically, if A, B, C, D are known, then the internal states x(t) can be computed in addition to the input u(t) and output y(t). Thus, one is effectively able to look inside the black box in Fig. 1.

The (internal) system poles (e.g. roots of Δ(s)) should be distinguished from the (input/output) poles of the transfer function. There may be some pole/zero cancellation in computing the transfer function. Then the denominator of H(s) is not the same as Δ(s), since some system poles do not occur in H(s). Therefore, the definitions of stability need to be modified to differentiate *internal stability* from the *external* (e.g. input/output stability) just defined.

The system is said to be:

(Internally) Asymptotically Stable (AS) if u(t)= 0 for all time t implies that x(t) goes to zero with time for all initial conditions x(0).

Bounded-Input/Bounded-Output Stable (BIBOS) if u(t) bounded for all time t implies that y(t) is bounded for all time when x(0)=0.

Though the first of these is usually simply called AS, it is not the same definition as the AS defined for input/output systems, where only the output y(t) is required to go to zero. Here, we require all the internal states to go to zero.

AS as defined here is concerned with the effects of the initial state x(0) on y(t), while BIBOS is concerned with the throughput effects of u(t) on y(t).

The system is (internally) AS iff all system natural modes decay with time. This occurs iff *all the system poles* are in the open-left half plane. For AS all the roots of $\Delta(s)$ must be in the OLHP.

To find a condition for BIBOS, recall that when x(0)= 0 the output may be found by convolving the impulse response h(t) with the input,

$$y(t) = \int_0^t h(t-t)u(t)\,dt \ .$$

If the impulse response is a decaying exponential, then the output is bounded for all bounded inputs. On the other hand, suppose h(t) is the unit step for instance. Convolution of any u(t) with the unit step simply gives the area under u(t). Thus, if u(t) is, e.g., also the unit step, then the output would be the unit ramp, which is not bounded. It turns out that the output y(t) is bounded for all bounded u(t) (when x(0)=0) if and only if the impulse response goes to zero with time.

Therefore, the system is BIBOS iff the poles of the transfer function are in the OLHP.

To study the AS of a system, one may apply the Routh test to the characteristic polynomial $\Delta(s)$. To study the BIBOS of a system, one may apply the Routh test to the denominator of the transfer function after pole/zero cancellation, if any.

Note: AS requires ALL the system poles to be in the OLHP. BIBOS only requires the poles remaining in the transfer function, after pole/zero cancellation, to be in the OLHP. Therefore, it is clear that AS implies BIBOS.

A single-input/single-output (SISO) system is said to be *minimal* if there is no pole/zero cancellation. In this event, the poles of H(s) are the same as the roots of Δ(s), so that the system is AS if and only if it is BIBOS. That is, for minimal systems, BIBOS also implies AS.

Example 6

Let $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x.$ Then $\Delta(s) = \begin{vmatrix} s & -1 \\ 1 & s+2 \end{vmatrix} = s^2 + 2s + 1 = (s+1)^2$ so that both poles are at s= -1. Therefore the system is AS. The natural modes are e^{-t}, te^{-t}.

Example 7

Let $\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = Cx = \begin{bmatrix} -1 & 1 \end{bmatrix} x$.

a. The characteristic polynomial is
$\Delta(s) = \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} = s^2 - 1 = (s+1)(s-1)$. The poles are at *s=-1, s=1*, so the system is not AS. It is unstable. The natural modes are e^{-t}, e^{t}.

b. The transfer function is
$H(s) = C(sI - A)^{-1} B = \frac{s-1}{(s-1)(s+1)} = \frac{1}{s+1}$,

which has poles at s=-1. Therefore, the system is BIBOS. Note that the unstable pole at s=1 has cancelled with a zero at s=1.