ELEC 3500
Control Systems
Lecture 7: Stability

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- **Stability of Input/Output Systems**
    - Asymptotically Stable
    - Marginally Stable
- **Routh Test**
- **Stability of State Variable Systems**
- (Internally) Asymptotically Stable
- Bounded-Input/Bounded-Output Stable (BIBOS)

❑ Stability of Input/Output Systems

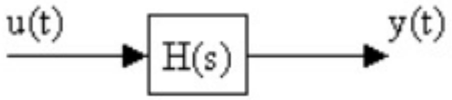
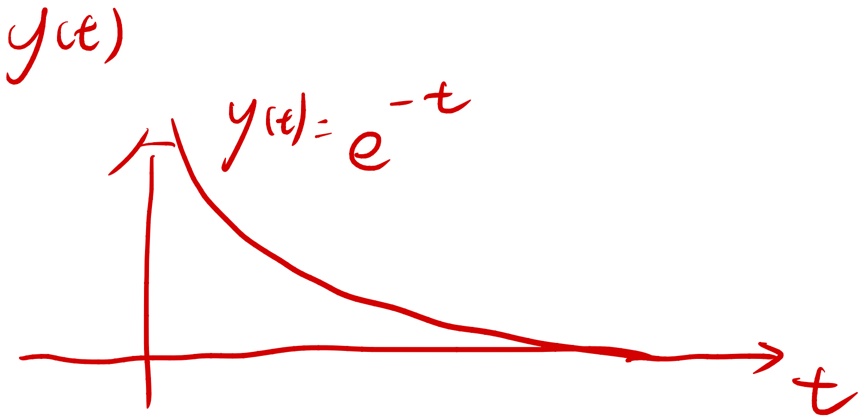
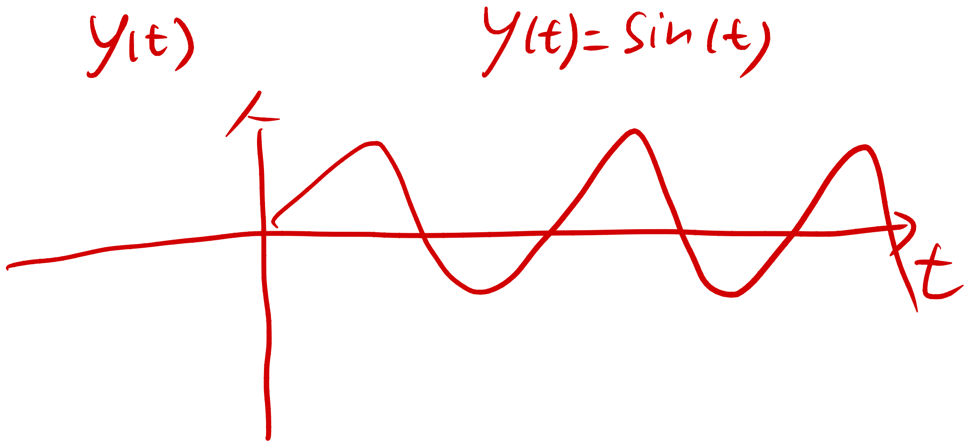


Figure 1



Asymptotically Stable, if $u(t) = 0$ for all time t implies that $y(t)$ goes to zero with time.

Marginally Stable, if $u(t) = 0$ for all time t implies that $y(t)$ is bounded for all time.



if only if

$$H(s) = \frac{N(s)}{\Delta(s)}$$

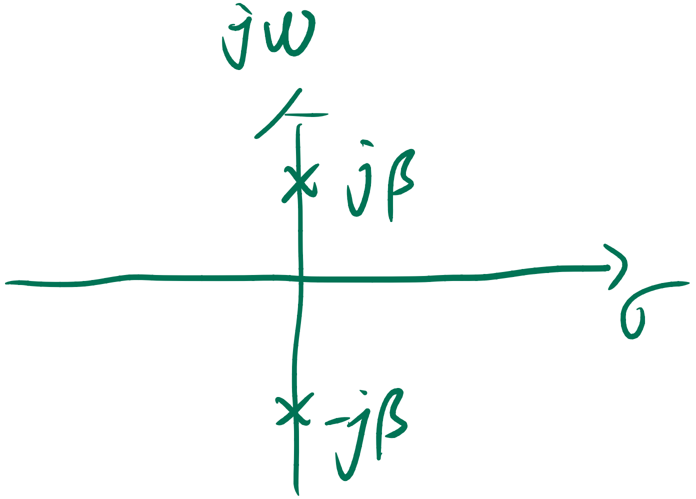
Check Principles:

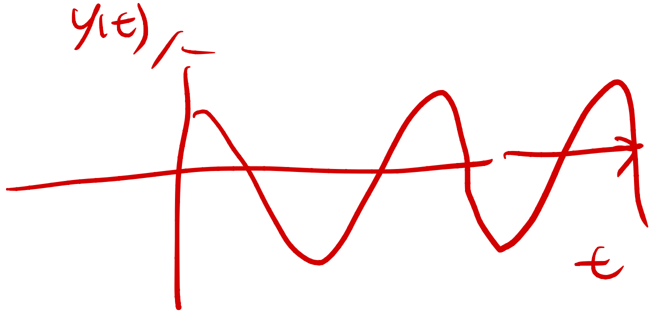
- The system is **Asymptotically Stable** iff all poles (roots of $\Delta(s)$) are in the open left-half plane (e.g., strictly in the left-half plane with none on the $j\omega$-axis).
- The system is **Marginally Stable** iff all poles (roots of $\Delta(s)$) are in the left-half plane (they may be in the open left-half plane or on the $j\omega$-axis), with any poles occurring on the $j\omega$-axis non-repeated.

$$y(t) = \cos \beta t$$

Examples:

$$H(s) = \frac{s}{s^2+\beta^2}$$





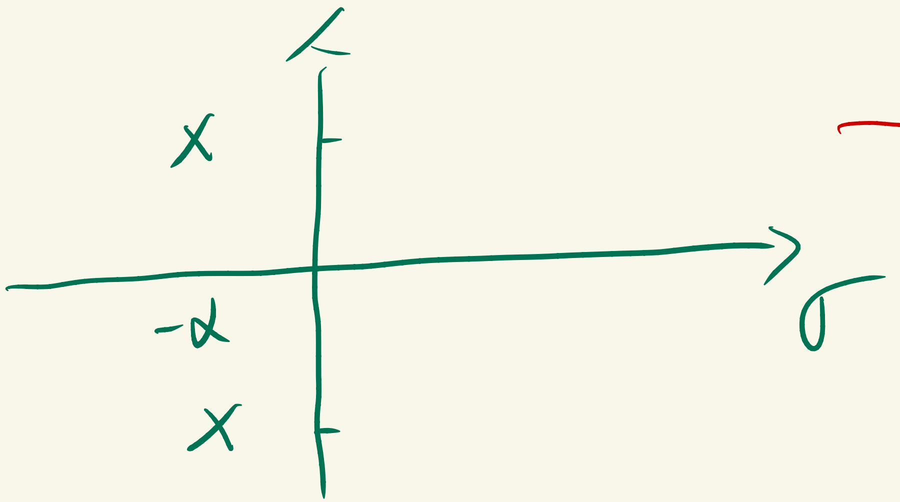
$$\Delta(s) = s^2+\beta^2 = 0$$

$$s = \pm j\beta \implies MS$$

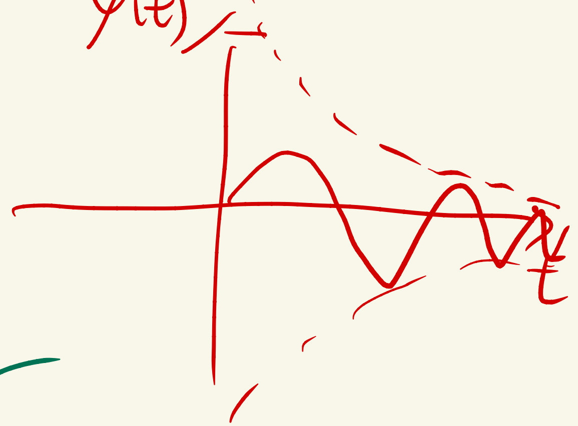
$$H(s) = \frac{\beta}{(s+\alpha)^2+\beta^2}$$

$\alpha > 0, \ \beta \neq 0$

poles: $-\alpha \pm j\beta$



$y(t) = e^{-\alpha t}\sin\beta t$



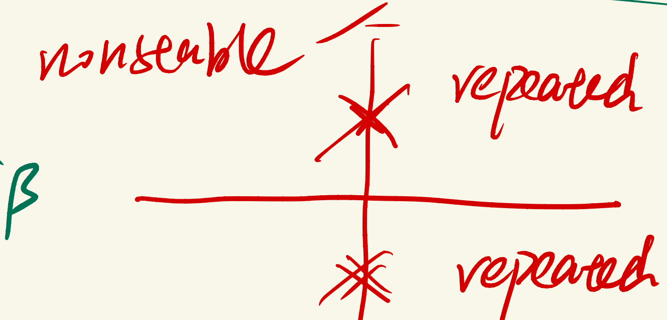
AS if $\alpha > 0, \ \beta = 0$

$-\alpha, \ -\alpha$

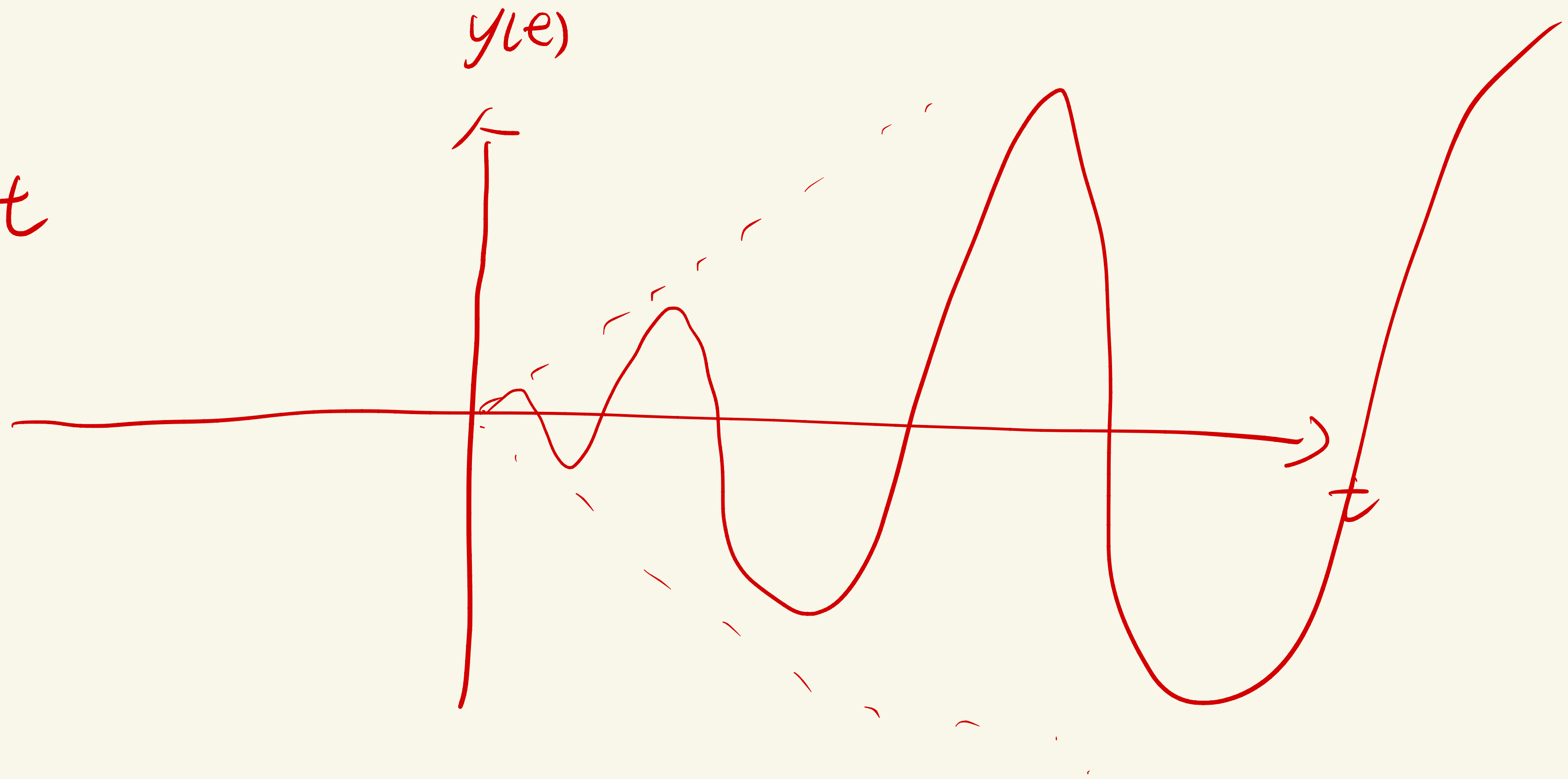
$$H(s) = \frac{s}{(s^2+\beta^2)^2}$$

y(t)

poles: $j\beta \quad -j\beta \quad j\beta \quad -j\beta$



$$y(t) = t\cos t$$



Routh Test

Given a polynomial *p(s)*, the number of poles in the right-half plane may be determined *without finding the roots* by using the Routh test.

$s^3 + s + 1$

Given $p(s) = a_0 s^n + a_1 s^{n-1} + \overset{a_2 s^{n-2}}{\cdots} + a_{n-1} s + a_n$, the Routh Array is given by

s^n	a_0	a_2	a_4	a_6 …
s^{n-1}	a_1	a_3	a_5	a_7 …
s^{n-2}	b_1	b_2	b_3	…
:	: c_1	: c_2	: c_3	
s^0				

$$b_1 = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}, \quad b_3 = \frac{-\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1}$$

$$c_1 = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1}$$

$$c_2 = \frac{-\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1}$$

The third row and below are each computed from the two rows immediately preceding it by using the matrix determinantal equations, e.g. for the third row.

Routh Theorem. The number of roots of *p(s)* in the right-half plane equals the number of sign changes in column one.

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & a_4 \end{array}$$

Example 1. $p(s) = s^4 + 2s^3 + 6s^2 + 10s + 7$

$p(s) = s^2 - \frac{7}{4}s - \frac{1}{2}$

Routh Array

s^4	a_0: 1	6	7	0
s^3	a_1: 2	10	0	
s^2	$b_1 = 1$	$b_2 = 7$		
s	$c_1 = -4$			
s^0	7			

$$-\begin{vmatrix} 1 & 6 \\ 2 & 12 \end{vmatrix} / 2 = -(10-12)/2 = 2/2$$

$$b_2 = -\begin{vmatrix} 1 & 7 \\ 2 & 0 \end{vmatrix} / 2 = -(0-14)/2 = 7$$

$$c_1 = -\begin{vmatrix} 2 & 10 \\ 1 & 7 \end{vmatrix} / 1 = -4$$

$\boxed{\text{2 poles in the right half plane}}$

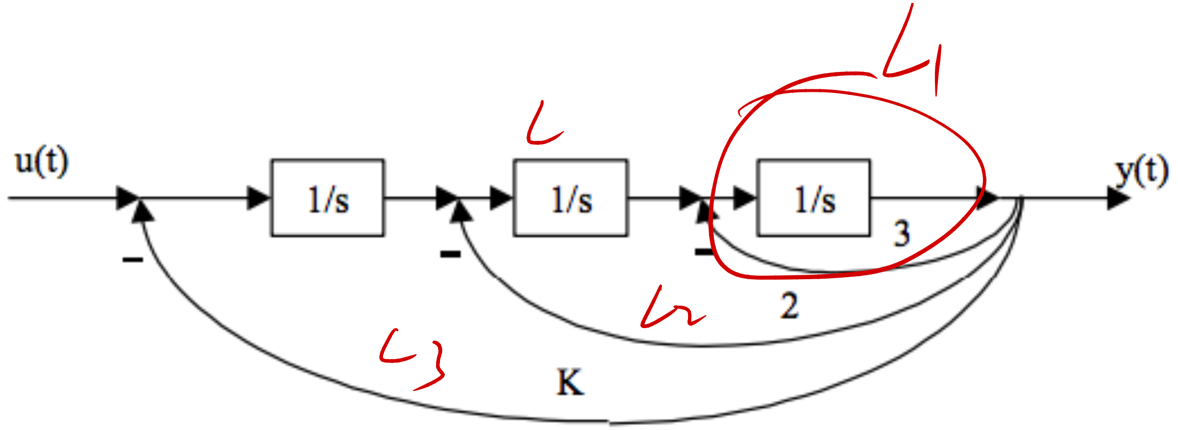
$$p(s) = s^2 - \frac{7}{4}s - \frac{1}{2}$$

Routh Array

$$\begin{array}{lll} s^2 & a_0 = 1 & -\frac{1}{2} \\ s & a_1 = -\frac{7}{4} & \\ s^0 & \boxed{-\frac{1}{2}} & \end{array}$$

sign changes once $\Rightarrow$ one pole in the right-half plane.

Example 2. In the figure, determine for what values of feedback gain K the system is stable.



step 1: Find $\Delta(s)$ in $H(s)$

$$H(s) = \frac{N(s)}{\Delta(s)}$$

$$\Delta(s) = 1 - \left(-\frac{3}{s} - \frac{2}{s^2} - \frac{K}{s^3} \right)$$

$$\Delta(s) = \frac{s^3 + 3s^2 + 2s + k}{s^3}$$ ☆

$$p(s) = \underset{a_0}{s^3} + \underset{a_1}{3s^2} + \underset{a_2}{2s} + \underset{a_3}{k} \quad a_4$$

Routh Array

$$\begin{array}{c|ccc}
 & & a_2 & a_4 \\
s^3 & 1 & 2 & 0 \\
 & a_1 & & \\
s^2 & 3 & k & \\
s & b_1 = & \frac{-\begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix}}{3} = \frac{6-k}{3} & \\
s^0 & \boxed{k} & &
\end{array}$$

$$\left\{ \begin{array}{l} \frac{6-k}{3} > 0 \\ k > 0 \end{array} \right. \Rightarrow 0 < k < 6$$

❑ Stability of State Variable Systems

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du$$

- **(Internally) Asymptotically Stable,** if $u(t) = 0$ for all time t implies that $x(t)$ goes to zero with time.
- **Bounded-Input/Bounded-Output Stable,** if $u(t)$ bounded for all time t implies that $x(t)$ is bounded for all time when $x(0)=0$.

Check Principles:

- The system is (internally) **Asymptotically Stable (AS)** iff all the system poles (roots of $|sI - A|$) are in the *open-left half plane (OLHP)*.
- The system is **Bounded-Input/Bounded-Output Stable (BIBOS)** iff the poles of the transfer function are in the *open-left half plane*.

Note:

- AS requires ALL the system poles to be in the OLHP, while BIBOS only requires the poles remaining in the transfer function, after pole/zero cancellation, to be in the OLHP.
- A single-input/single-output system is ***minimal*** if there is no pole/zero cancellation. In this event, the poles of *H(s)* are the same as the roots of $\Delta(s)$, so that the system is AS if and only if it is BIBOS. That is, for minimal systems, BIBOS also implies AS.

Example 1 $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x$

Example 2 $\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = Cx = \begin{bmatrix} -1 & 1 \end{bmatrix} x$