ELEC 3500
Control Systems
Lecture 6: State Variable Analysis

Instructor: Dr. Bosen Lian
Electrical and Computer Engineering
Auburn University
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- **State Variable (SV) Analysis**
  - Frequency-Domain Solution of SV Systems
  - Time-Domain Solution of SV Systems
- **Relative Degree**
- **Zeros of State-Space Systems**
- **Realization and Canonical Forms**

- Find a SV or block diagram representation given a prescribed transfer function
- Reachable Canonical Form (RCF)
- Observable Canonical Form (OCF)

$$\dot{x} = Ax + Bu$$

LTI

$$y = Cx + Du$$

$x \in \mathbb{R}^n$ internal state, $u(t) \in \mathbb{R}^m$ the control input

$y(t) \in \mathbb{R}^p$ the measured output

A the plane/system

B the control input matrix

C output matrix

D direct feedforward matrix

State Variable Analysis

- Frequency-Domain Solution of SV Systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Laplace Transform

$sX(s) - X(0) = AX(s) + BU(s)$

$Y(s) = CX(s) + DU(s)$

$sX(s) - AX(s) = X(0) + BU(s)$

$X(s) = (sI - A)^{-1}X(0) + (sI-A)^{-1}BU(s)$

$Y(s) = C(sI - A^{-1})X(0) + \boxed{C(sI-A)^{-1}B + D}\,U(s)$

transfer function $H(s)$

$H(s)U(s) = Y(s)$

- Time-Domain Solution of SV Systems

$$y(t) = L^{-1}(C(SI - A)^{-1})x(0) + L^{-1}\{[C(SI - A)^{-1}B + D]U(s)\}$$

- Resolvent matrix

$$\Phi(s) = (SI - A)^{-1}$$

$$Y(s) = \underbrace{C(sI-A)^{-1}x(0)}_{\text{intial terms}} + \underbrace{\left(C(sI-A)^{-1}B + D\right)U(s)}_{\text{depends on conrol input}}$$

$$Y(s) = H(s)\,U(s)$$

$$H(s) = C(sI-A)^{-1}B + D$$

denominator of this is the determinant of $(sI-A)$, denoted by

$$\Delta(s) = |sI-A|$$

roots are system poles $\Delta(s) = |sI - A| = 0$

Resolvent matrix

$$\Phi(s) = (sI - A)^{-1}$$

$$\boxed{\begin{aligned} X(s) &= \underline{\Phi}\,\underline{X(0)} + \underline{\Phi} B \underline{U} \\ Y(s) &= C \Phi X(0) + (C \Phi B + D) U \\ H(s) &= C \Phi B + D \end{aligned}}$$

Time domain solution

$$x(t) = \mathcal{L}^{-1}(X(s))$$

$$y(t) = \mathcal{L}^{-1}(Y(s))$$

$u(t) = e^{-3t}$

$\downarrow$

$u(s) = \frac{1}{s+3}$

step response: Let $U(s) = \frac{1}{s}$

$$Y(s) = C\Phi x(0) + H(s)\cdot\frac{1}{s}$$

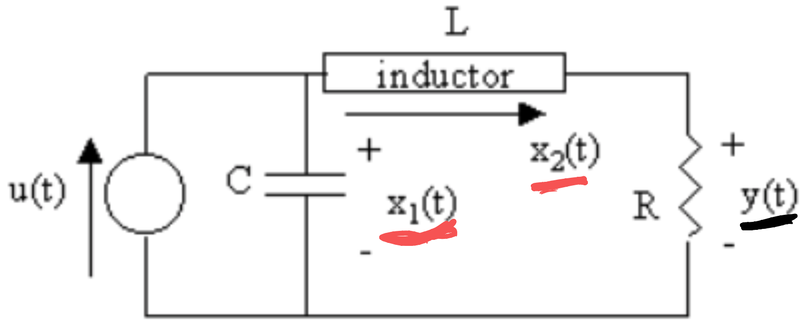
$$y(t) = \mathcal{L}^{-1}(Y(s))$$

impulse response: Let $U(s) = 1$

$$y(t) = \mathcal{L}^{-1}(C\Phi x(0) + \underline{H(s)})$$

· state-variable systems

Example 1: Electrical Circuit



→ output

$$C\frac{dx_1}{dt} + x_2 = u(t)$$

$$L\frac{dx_2}{dt} + R x_2 = x_1$$

$$\dot{x}_1 = -\frac{x_2}{C} + \frac{u(t)}{C}$$

$$\dot{x}_2 = -\frac{R}{L}x_2 + \frac{x_1}{L}$$

Solve for

- State Equation
- Frequency Domain
- Time Constant, Natural Frequency, etc
- Time Domain with zero initials given L= 1 h, R= 3 Ω, C= 0.5 f

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -\frac{1}{C}\\ \frac{1}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{C}\\ 0\end{bmatrix}u(t)$$

$$y(t) = R\cdot x_2 = [0 \quad R]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Now we have

$$\dot{X} = AX + BU \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Y = CX$$

where $A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$

$$C = \begin{bmatrix} 0 & R \end{bmatrix}$$

- Frequency domain

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s+\frac{R}{L} \end{vmatrix} = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

$$\Phi(s) = (sI-A)^{-1} = \begin{pmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s+\frac{R}{L} \end{pmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s+\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

$$H(s) = C(sI-A)^{-1}B + \cancel{D} = \frac{[0 \quad R] \begin{bmatrix} s+\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}}{\Delta(s)}$$

$$= \frac{\begin{bmatrix} \frac{R}{L} & R\cdot s \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}}{\Delta(s)}$$

$$= \frac{R}{LC} \cdot \frac{1}{\Delta(s)} = \frac{\frac{R}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \ //$$

- Time constant, natural frequency

$$\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

$$= s^2 + 2\alpha s + \underline{w_n}^2 = s^2 + 2\zeta w_n s + w_n^2$$

decay term $\alpha = \frac{R}{2L}$

time constant $\tau = \frac{1}{\alpha} = \frac{2L}{R}$

natural frequency $\omega_n = \sqrt{\frac{1}{LC}}$

damping ratio $\zeta = \frac{\alpha}{\omega_n} = \frac{R}{2}\sqrt{\frac{C}{L}}$

d. Time domin solution

select $L = 1$, $R = 3$, $C = 0.5$

$$H(s) = \frac{6}{s^2 + 3s + 2}$$

impluse response with zero intial condition

$x(0) = 0$

$$Y(s) = H(s) \cdot U(s)$$

$$U(s) = 1$$

$$Y(s) = H(s)$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}(H(s))$$

$$H(s) = \frac{6}{s^2+3s+2} = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$K_1 = (s+1)\frac{6}{s^2+3s+2}\Big|_{s=-1} = \frac{6}{1} = 6$$

$$K_2 = (s+2)\frac{6}{s^2+3s+2}\Big|_{s=-2} = \frac{6}{-1} = -6$$

$$H(s) = \frac{6}{s+1} + \frac{-6}{s+2}$$

$$y(t) = \mathcal{L}^{-1}(H(s))$$
$$= \left(6e^{-t} - 6e^{-2t}\right)u_{-1}(t)$$

Example 2: Newton's third law

$$F = ma$$

$$\ddot{y} = \frac{F}{m} \equiv u(t)$$

$u(t)$ the force per unit mass or acceleration

$x = [x_1 \quad x_2]^T$

$$x_1 = y$$

$$x_2 = \dot{y}$$

then

$$\dot{x}_1 = \dot{y}$$

$$\dot{x}_2 = \ddot{y} = u$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Transfer function

$$H(s) = C(sI-A)^{-1}B + \cancel{D}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0] \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{[s \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2} = \frac{1}{s^2}$$

$$\Delta(s) = |sI - A| = 0$$

$$s^2 = 0$$

poles : $0, \ 0$

$\downarrow \qquad \downarrow$

natural modes : $u_{-1}(t), \ \underline{t u_{-1}(t)}$

Find state response given $x(0) = [\, s_0 \quad v_0 \,]^T$ and input

$u = g$,

$U(s) = \frac{g}{s}$

$$X(s) = \Phi(s)\, x(0) + \Phi B U$$

$$X(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} s_0 \\ v_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{g}{s}$$

$$= \begin{bmatrix} \frac{s_0}{s} + \frac{v_0}{s^2} + \frac{g}{s^3} \\ \frac{v_0}{s} + \frac{g}{s^2} \end{bmatrix}$$

$$x(t) = \mathcal{L}^{-1}\left(X(s) \right)$$

$$= \begin{bmatrix} s_0 + v_0 t + \frac{1}{2} g t^2 \\ v_0 + g t \end{bmatrix} u_{-1}(t)$$

❑ Relative Degree

$$H(s) = C\Phi(s)B + D = C(sI - A)^{-1}B + D$$
$$= \frac{C[adj(sI - A)]B}{|sI - A|} + D = \frac{C[adj(sI - A)]B + D|sI - A|}{|sI - A|} = \frac{N(s)}{\Delta(s)}$$

The *relative degree* of *H(s)* is the degree of the denominator minus the relative degree of the numerator.

If $A \in R^{n \times n}$

$n - (n-1) \geq 1$, D is 0

$n - n = 0$, D is not 0

❑ Zeros of SV Systems

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} a & b \end{bmatrix} x$$

$D = 0$, $\Phi(s) = (sI - A)^{-1}$

$$H(s) = C \cdot \Phi(s) B + D$$

$$= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{bs + a}{(s+1)(s+2)}$$

$z_f \quad b/a = 1$, $H(s) = \frac{a \cdot \cancel{(s+1)}}{\cancel{(s+1)}(s+2)}$

$z_f \quad a/b = 2$, $(s+2)$ is cancelled,

Realization Problem

Transfer function

SV systems

BD

Realization and Canonical Forms

- Reachable Canonical Form (RCF)

$b_3 s^3$

$g_1 \Delta_1 + g_2 \Delta_2 + g_2 \Delta_3$

$$H(s)=\frac{b_2 s^2+b_1 s+b_0}{s^3+a_2 s^2+a_1 s+a_0} \quad /s^3 \;\Rightarrow\; H(s)=\frac{b_2 s^{-1}+b_1 s^{-2}+b_0 s^{-3}}{1+a_2 s^{-1}+a_1 s^{-2}+a_0 s^{-3}}=\boxed{\frac{b_2 s^{-1}+b_1 s^{-2}+b_0 s^{-3}}{1-(-a_2 s^{-1}-a_1 s^{-2}-a_0 s^{-3})}}.$$

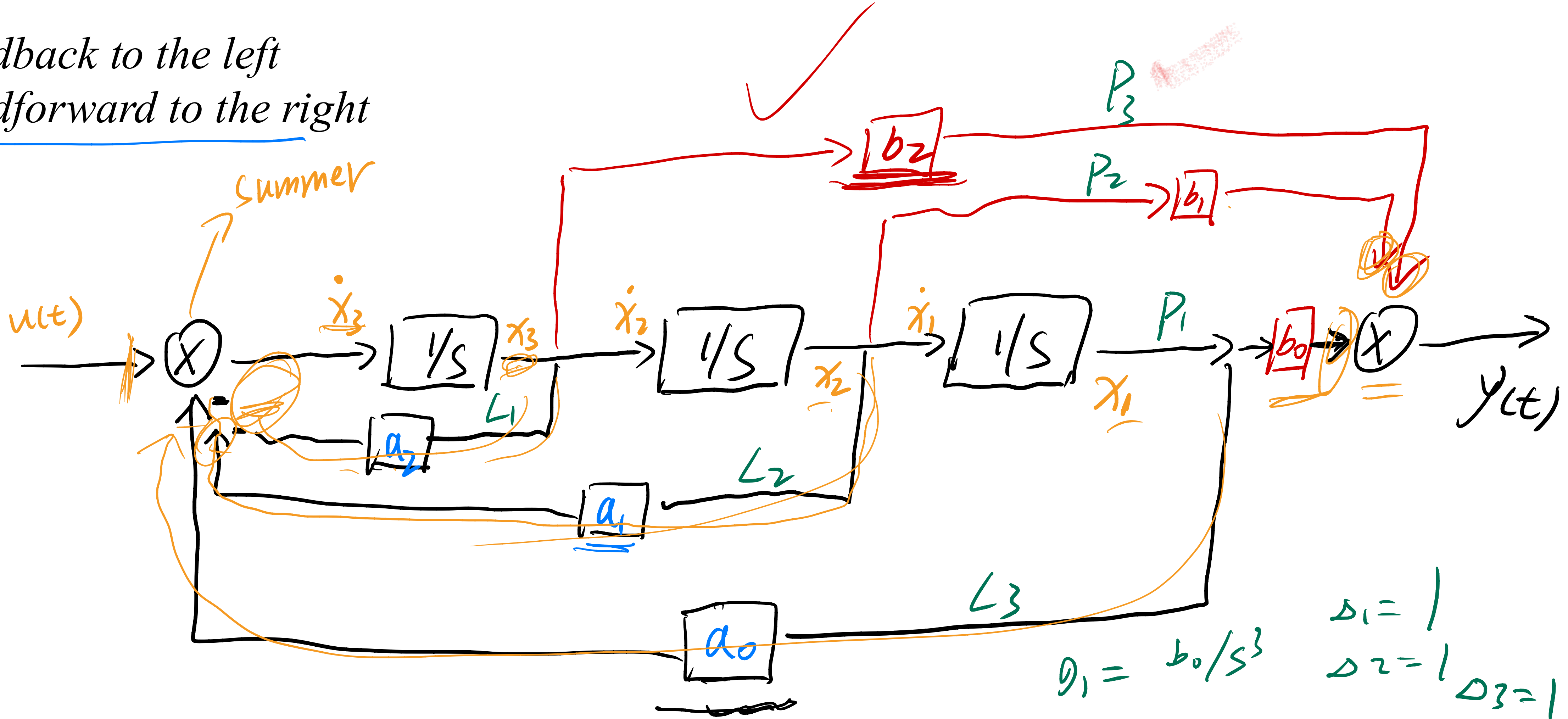
1 • Ensure that the transfer function is *monic*, that is, the highest order term in the denominator has a coefficient of 1. If not, divide through by this coefficient to put the transfer function in monic form.

2 • Ensure that transfer function must also have relative degree of 1 or more. If not, write *H(s)* as a constant term plus a term whose relative degree is at least one.

1 − transmission of all loops

The rule used for RCF is

feedback to the left
feedforward to the right



$\Delta_1 = 1$

$g_1 = b_0/s^3 \quad \Delta_2 = 1 \quad \Delta_3 = 1$

$$\Delta(s) = 1 - \left(- a_2/s - a_1/s^2 - a_0/s^3\right)$$

$g_2 = b_1/s^2 \quad g_3 = \frac{b_2}{s}$

❑ Realization and Canonical Forms

- Find SV representation given a prescribed transfer function

$\dot{x} \longrightarrow \boxed{1/s} \longrightarrow x \iff x = \int \dot{x}$

Each integrator output is labeled as a state. The rule for labeling states will be:

Label the states from right to left

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + u(t)$

$y = b_0 x_1 + b_1 x_2 + b_2 x_2$

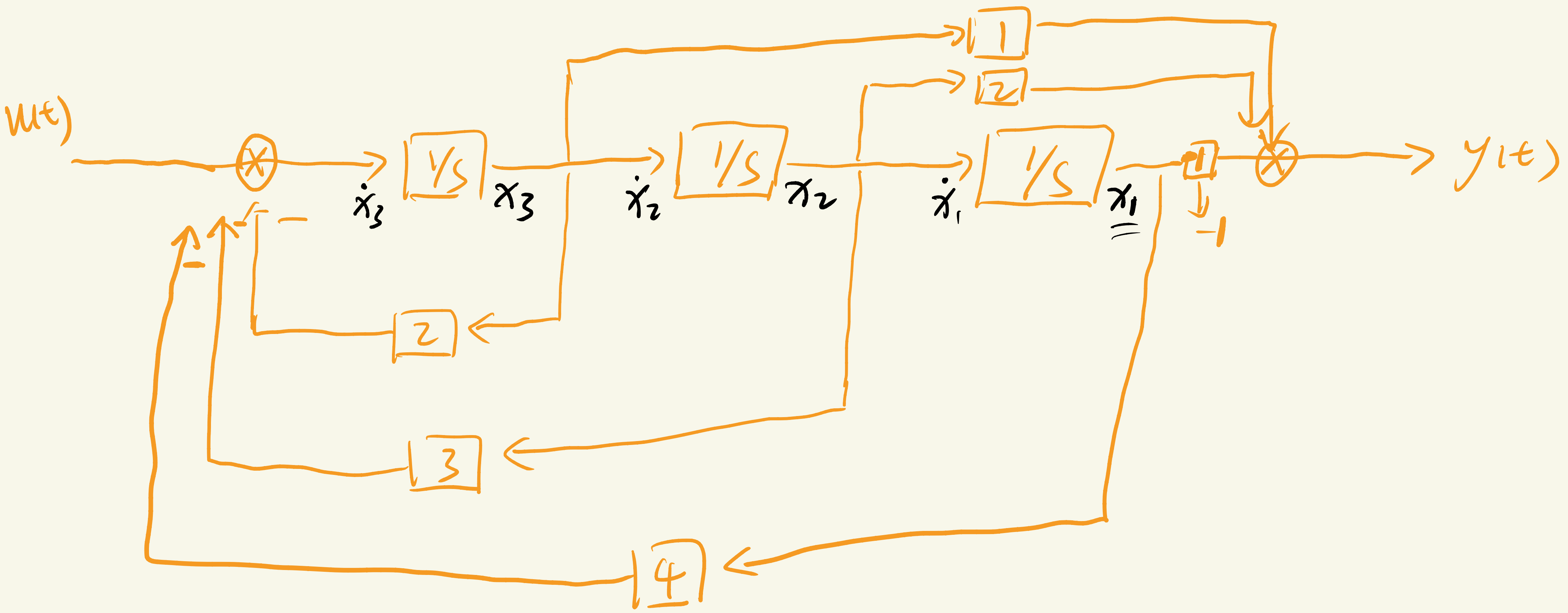
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 1

$$H(s) = \frac{s^2 + 2s - 1}{s^3 + 2s^2 + 3s + 4}.$$

$$H(s) = \frac{(s^2+2s-1)/s^3}{(s^3+2s^2+3s+4)/s^3} = \frac{s^{-1}+2s^{-2}-s^{-3}}{1-(-2s^{-1}-3s^{-2}-4s^{-3})}$$



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 \qquad \Rightarrow \quad \dot{x} = Ax + Bu$$

$$\dot{x}_3 = -4x_1 - 3x_2 - 2x_3 + u(t)$$

$$y(t) = -x_1 + 2x_2 + x_3 \qquad \Rightarrow \quad y = Cx$$

❑ Realization and Canonical Forms

- Observable Canonical Form (OCF)

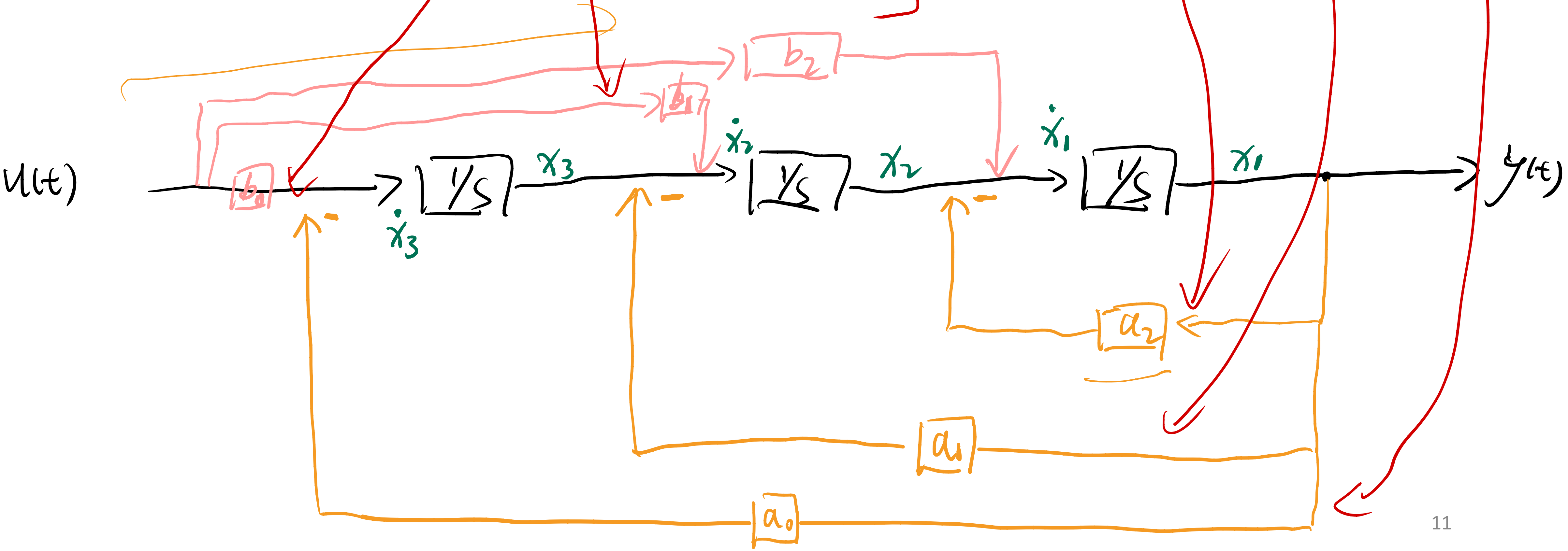
$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \Longrightarrow H(s) = \frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}} = \boxed{\frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 - (-a_2 s^{-1} - a_1 s^{-2} - a_0 s^{-3})}}.$$

- Ensure that the transfer function is *monic*, that is, the highest order term in the denominator has a coefficient of 1. If not, divide through by this coefficient to put the transfer function in monic form.
- Ensure that transfer function must also have relative degree of 1 or more. If not, write *H(s)* as a constant term plus a term whose relative degree is at least one.

The rule used for OCF is

feedback from the right
feedforward from the left

$$H(s) = \frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 - (a_2 s^{-1} - a_1 s^{-2} - a_0 s^{-3})}$$



❑ Realization and Canonical Forms

- Find SV representation given a prescribed transfer function

Each integrator output is labeled as a state. The rule for labeling states will be:

Label the states from bottom to top

from right to left

$$\dot{x}_1 = -a_2 x_1 + x_2 + b_2 u(t)$$

$$\dot{x}_2 = -a_1 x_1 + x_3 + b_1 u(t)$$

$$\dot{x}_3 = -a_0 x_1 + b_0 u(t)$$

$$y(t) = x_1(t)$$

Example 2

1 2 −1

$$H(s) = \frac{s^2 + 2s - 1}{s^3 + 2s^2 + 3s + 4}$$

2

$$\begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$