BLOCK DIAGRAM AND MASON'S FORMULA

A linear time-invariant (LTI) system can be represented in many ways, including:

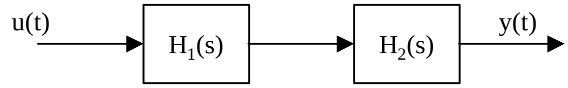
- differential equation
- state variable form
- transfer function
- impulse response
- block diagram or flow graph

Each description can be converted to the others. In this lecture we shall see how to represent systems in terms of block diagrams, and how to determine the transfer function of a block diagram system using Mason's Formula.

SYSTEM INTERCONNECTIONS

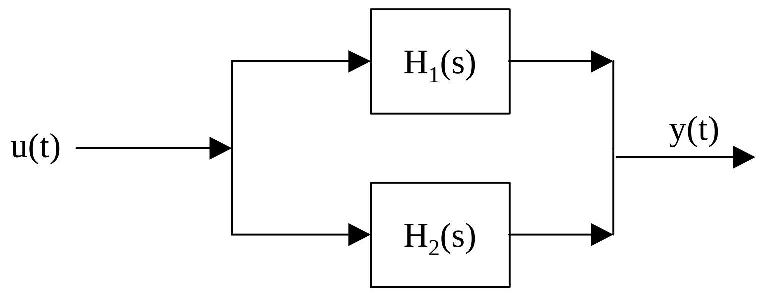
Systems can be interconnected in a variety of ways, including the following.

Series Interconnection



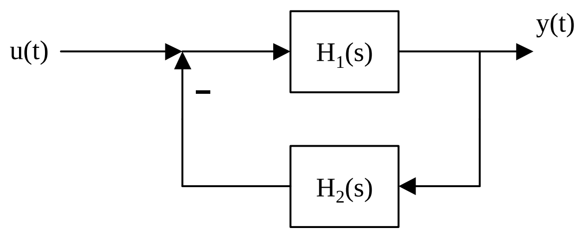
The overall transfer function in $Y(s) = H(s)U(s)$ is given by the product $H(s) = H_1(s)H_2(s)$.

Parallel Interconnection



The overall transfer function in $Y(s) = H(s)U(s)$ is given by the sum $H(s) = H_1(s) + H_2(s)$.

Feedback Interconnection



The overall transfer function in $Y(s) = H(s)U(s)$ is given by

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}.$$

We shall see later how to derive this using Mason's Formula.

Example 1- Transfer Function of Feedback Configuration

An unstable plant has transfer function

$$H_1(s) = \frac{1}{s-1}$$

which has one pole in the right-half plane at *s=1*. To stabilize the plant, one may use a feedback configuration with compensator

$$H_2(s) = \frac{10(s+2)}{s+5}.$$

The closed-loop transfer function is

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{\frac{1}{s-1}}{1 + \left(\frac{1}{s-1}\right)\left(\frac{10(s+2)}{s+5}\right)} = \frac{s+5}{(s-1)(s+5) + 10s + 20}$$

$$= \frac{s+5}{s^2 + 14s + 15} = \frac{s+5}{(s+1.17)(s+12.83)}$$

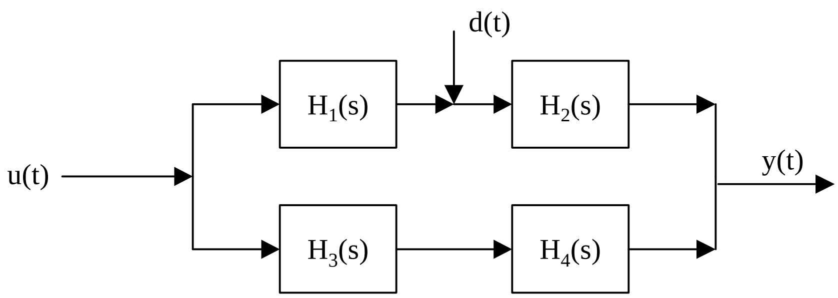
This is stable with two left-half plane poles. The compensator has stabilized the plant.

Superposition for Linear Systems

For LTI systems *superposition* holds, so that the effects of different inputs can simply be added together. The formula above can be applied several times and mixed together to derive the transfer function in many systems.

Example 2- Superposition

In this system, *u(t)* might represent the control input while *d(t)* is a disturbance input.



The system can be viewed as two systems, one driven by input *u(t)* and one driven by input *d(t)*. Setting to zero *u(t)* one can determine that

$Y(s) = H_2(s)D(s)$.

Setting to zero *d(t)*, one can consider the transfer from *u(t)* to *y(t)* as two parallel branches of systems in series. Thus, the formulae above yield

$Y(s) = [H_1(s)H_2(s) + H_3(s)H_4(s)]U(s)$.

Since the system is LTI, the overall output is the sum (superposition) of the effects of the two inputs

$Y(s) = [H_1(s)H_2(s) + H_3(s)H_4(s)]U(s) + H_2(s)D(s)$.

One can define two transfer functions to capture this. If

$$H_Y(s) \equiv H_1(s)H_2(s) + H_3(s)H_4(s)$$
$$H_D(s) = H_2(s)$$

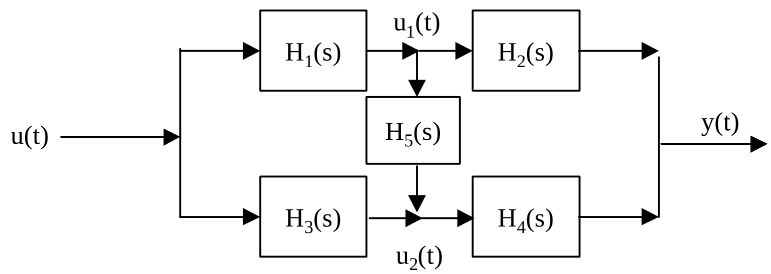
then

$$Y(s) = H_Y(s)Y(s) + H_D(s)D(s)\,.$$

This is a very simple example with no feedback loops. In more complex situations one must use Mason's Formula to evaluate the overall transfer function.

Example 3- Superposition

It is easy to take into account multiple feedforward paths if there are no feedback loops.



One may write

$$U_1 = H_1 U$$
$$U_2 = H_3 U + H_5 U_1 = H_3 U + H_5 H_1 U$$
$$Y = H_2 U_1 + H_4 U_2 = H_2 H_1 U + H_4 (H_3 U + H_5 H_1 U)$$
$$Y = (H_2 H_1 + H_4 H_3 + H_4 H_5 H_1) U$$

MASON'S FORMULA

Mason's Formula allows one to determine the transfer function of general block diagrams with multiple loops, including feedback loops, and multiple feedforward paths. It relies on some ideas that we now define.

A block diagram consists of *paths* and *loops*. A *loop* is any path where one can go in a circle and return to the beginning point by following arrows in the direction in which they point. Note that the figures in Examples 2 and 3 do not contain any loops. Two loops are said to be *disjoint* if they have no elements in common, i.e. if they do not touch.

The *determinant* of a block diagram is defined as

D(s)=1 - (sum of transmissions of all loops)
+ (sum of products of transmissions of all pairs of disjoint loops)
- (sum of products of transmissions of all triples of disjoint loops)
- + …

The *cofactor* of a block diagram with respect to the *i*-th path is defined as

D_i(s)=1 - (sum of transmissions of all loops that are disjoint from path i)
+ (sum of products of transmissions of all pairs of disjoint loops that are disjoint from path i)
- (sum of products of transmissions of all triples of disjoint loops that are disjoint from path i)
+ …

The *i*-th cofactor is the same as the determinant, but does not include any loops touching path *i*.

Also required is

g_i= transmission along path i

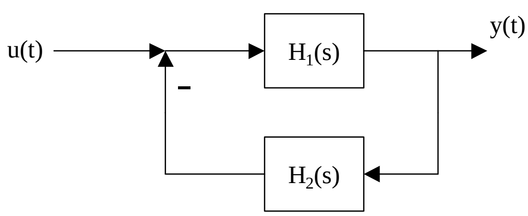
In general, these are all functions of *s* so that one writes $\Delta(s), \Delta_i(s), g_i(s)$. In terms of the constructions, one may write the transfer function of any block diagram as

Mason's Formula

$$H(s) = \frac{1}{\Delta(s)} \sum_i g_i(s) \Delta_i(s)$$

It is intriguing to realize that this is simply the formula for the inverse of a matrix in terms of the cofactors in a Laplace expansion.

Example 4- Feedback Connection Using Mason's Formula



Let us use Mason's Formula to determine the transfer function of the Feedback Interconnection.

There is one loop, which has loop transmission $-H_1H_2$. Therefore, the determinant is

$$\Delta(s) = 1 - \quad loop\ gain = 1 + H_1H_2 .$$

There is one path from input to output, which has transmission

$$g_1 = H_1 .$$

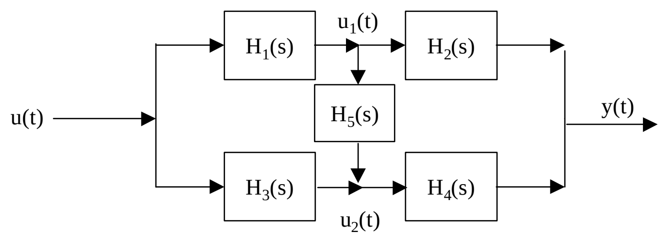
The loop touches the path, so there are no loops disjoint from the path. Therefore one has $\Delta_1 = 1$.

Applying now Mason's formula one obtains

$$H(s) = \frac{1}{\Delta(s)} g_1(s)\Delta_1(s)$$

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} .$$

Example 5- Redo Example 3 Using Mason's Formula



There are no loops so the determinant is $\Delta(s) = 1$. There are three feedforward paths from input to output. One has

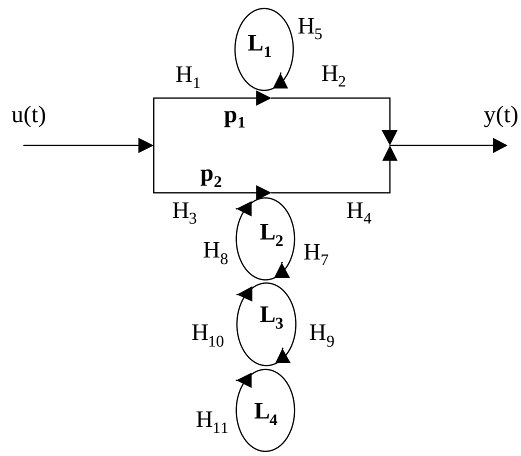
$$g_1 = H_1H_2$$
$$g_2 = H_1H_5H_4 .$$
$$g_3 = H_3H_4$$

There are no loops disjoint from the paths, so all cofactors are equal to 1. Therefore, Mason's Formula reveals that

$$H(s) = g_1 + g_2 + g_3 = H_2H_1 + H_4H_3 + H_4H_5H_1 .$$

Example 6- Multiloop System

Even complicated multiloop systems are susceptible to straightforward solution using Mason's Formula.



There are two paths from input to output and four loops, all labeled. Note that one could manufacture another path from *u(t)* to *y(t)* by going along part of $\mathbf{p_1}$, then circuiting loop $\mathbf{L_1}$, then completing path $\mathbf{p_1}$. However, we consider only independent paths and loops, not direct compositions of simpler paths and loops.

The pairs of disjoint loops are ($\mathbf{L_1,L_2}$), ($\mathbf{L_1,L_3}$), ($\mathbf{L_1,L_4}$), ($\mathbf{L_2,L_4}$). The triple ($\mathbf{L_1,L_2,L_4}$) is disjoint. Therefore the determinant is

$$\begin{aligned}\Delta(s) = 1 &- (H_5 + H_7H_8 + H_9H_{10} + H_{11}) \\ &+ (H_5H_7H_8 + H_5H_9H_{10} + H_5H_{11} + H_7H_8H_{11}) . \\ &- (H_5H_7H_8H_{11})\end{aligned}$$

The two paths have transmissions

$$\begin{aligned} g_1 &= H_1H_2 \\ g_2 &= H_3H_4 \end{aligned} .$$

It is easy to determine the cofactors. To find the cofactor of path 1, simply go to the formula for the determinant and cross out all loops touching path 1. Loop $\mathbf{L_1}$ touches path 1, so we cross out all terms containing H_5 to obtain

$$\begin{aligned}\Delta_1(s) = 1 &- (H_7H_8 + H_9H_{10} + H_{11}) \\ &+ (H_7H_8H_{11})\end{aligned} .$$

To find the cofactor of path 2, simply go to the formula for the determinant and cross out all loops touching path 2. Loop $\mathbf{L_2}$ touches path 2, so we cross out all terms containing H_7H_8 to obtain

$$\begin{aligned}\Delta_2(s) = 1 - (H_5 + H_9 H_{10} + H_{11}) \\ + (H_5 H_9 H_{10} + H_5 H_{11})\end{aligned}.$$

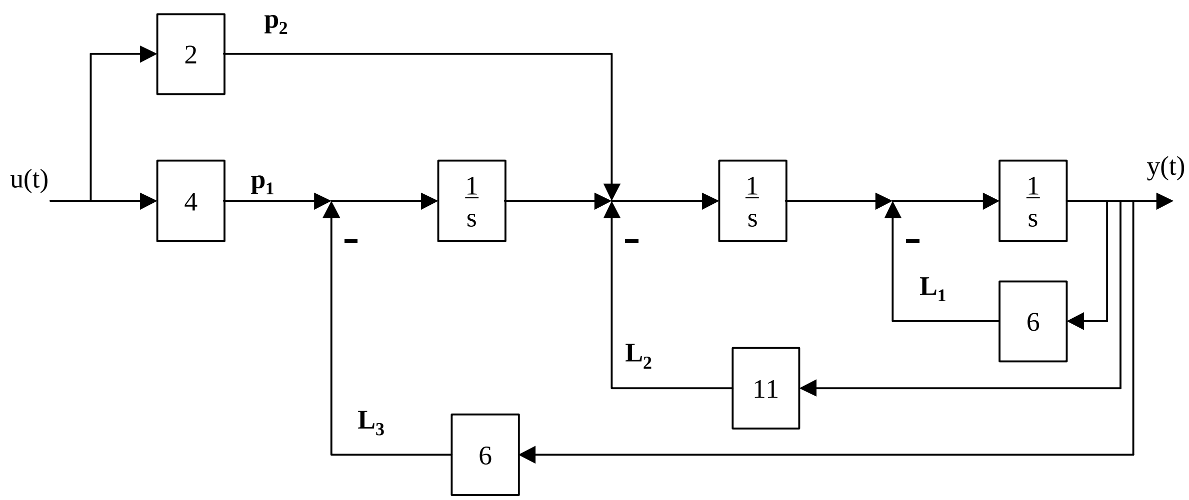
The transfer function is now given by Mason's Formula

$$H = \frac{H_1 H_2 \Delta_1 + H_3 H_4 \Delta_2}{\Delta}.$$

For any specific example given, all the transfer functions will be prescribed (c.f. Example 1), and one substitutes into this equation and simplifies to find *H(s).* Then one may find the step response, poles, zeros, output given any input, etc.

Example 6- System in Observable Canonical Form

a. Transfer Function



This block diagram is in *observable canonical form,* and is typical of many we shall see in analyzing state space systems. We shall see later that the output of each integrator is a state. There are 3 loops and 2 feedforward paths. All loops touch, and each loop touches both paths. In fact, all loops and paths contain the rightmost integrator.

The determinant is

$$\Delta = 1 - (\frac{-6}{s} - \frac{11}{s^2} - \frac{6}{s^3}) = \frac{s^3 + 6s^2 + 11s + 6}{s^3}.$$

The path transmissions are

$$\begin{aligned} g_1 &= \frac{4}{s^3} \\ g_2 &= \frac{2}{s^2} \end{aligned}.$$

Both cofactors are equal to 1 as there are no loops disjoint from either path.

The transfer function is given by

$$H(s) = \frac{g_1 + g_2}{\Delta} = \frac{\left(\frac{4}{s^3} + \frac{2}{s^2}\right)}{\left(\frac{s^3 + 6s^2 + 11s + 6}{s^3}\right)} = \frac{2s+4}{s^3 + 6s^2 + 11s + 6}.$$

Note that the denominator of *H(s)* is the characteristic polynomial which we by denote *D(s)*. The quantity denoted by *D(s)* in Mason's Formula is the determinant, which is the characteristic polynomial divided by the highest power of *s*.

Note that

$$H(s) = \frac{2(s+2)}{(s+1)(s+2)(s+3)} = \frac{1}{(s+1)(s+3)}$$

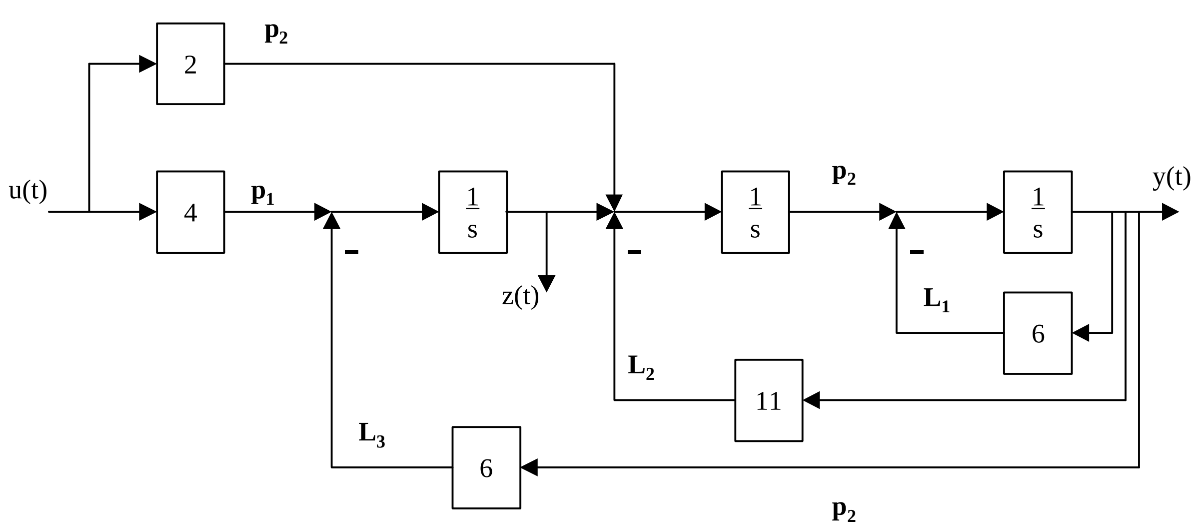
so there is pole/zero cancellation. This system is actually of second order, and it can in fact be realized using two integrators instead of three. The block diagram is said to be *nonminimal* since it realizes its transfer function using too many integrators.

Noe that if the feedforward gain in path $\mathbf{p_2}$ is changed from 2 to 1, then one has

$$H(s) = \frac{s+4}{(s+1)(s+2)(s+3)}$$

where there is no pole/zero cancellation. This system is of order three and the block diagram realization is minimal.

b. Different Output



Different inputs and outputs can be defined for the same block diagram. Let us find the transfer function from *u(t)* to the new output *z(t)* shown in the figure.

Selecting new inputs or outputs does not change the basic structure of the block diagram, so the loop structure is the same and the determinant is the same.as in part a.

Though it is not obvious at first glance, there are two paths from *u(t)* to *z(t)*. They are labeled in the figure. Path $\mathbf{p_2}$ is a circuitous journey consisting of the elements 2, 1/s, 1/s, 6, -,1/s. The path transmissions are

$$g_1 = \frac{4}{s}$$
$$g_2 = \frac{-12}{s^3}.$$

All loops touch path $\mathbf{p_2}$ so that $D_2=1$. However, loops $\mathbf{L_1}$ and $\mathbf{L_2}$ are disjoint from path $\mathbf{p_1}$ so that

$$\Delta_1(s) = 1 - \left(-\frac{6}{s} - \frac{11}{s^2} \right) = \frac{s^2 + 6s + 11}{s^2}.$$

According to Mason's Formula one has the transfer function from *u(t)* to *z(t)* given by

$$H_{zu}(s) = \frac{g_1\Delta_1 + g_2\Delta}{\Delta} = \frac{\frac{4}{s}\left(\frac{s^2 + 6s + 11}{s^2}\right) + \frac{-12}{s^3}}{\left(\frac{s^3 + 6s^2 + 11s + 6}{s^3}\right)} = \frac{4(s^2 + 6s + 8)}{s^3 + 6s^2 + 11s + 6}.$$

Note that

$$H_{zu}(s) = \frac{4(s+2)(s+4)}{(s+1)(s+2)(s+3)} = \frac{4(s+4)}{(s+1)(s+3)},$$

so the block diagram is nonminimal with respect to the I/O pair *(u(t),z(t))*.

MINIMALITY, POLES, AND ZEROS

A block diagram is said to be *minimal* if it realizes its transfer function with the minimum number of integrators. For single-input/single-output (SISO) systems, this occurs if and only if there is no pole/zero cancellation. Note that the same block diagram can be minimal with respect to one input/output (I/O) pair but nonminimal with respect to another.

The poles are determined by the loops and the zeros by the feedforward paths. Note that the zeros change as the input/output pair is changed, but the poles depend on the basic loop structure and are independent of the selection of inputs and outputs.

Once Mason's Formula has been used to determine the transfer function, one may determine for any block diagram the step response, poles, natural modes, output given a prescribed input, and so on.