ELEC 3500
Control Systems
Lecture 4: Transfer Function, Poles, Zeros, Response

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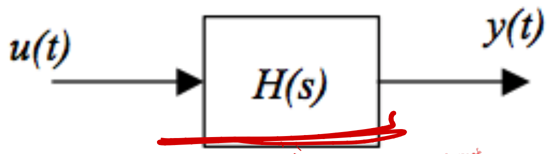
Fall 2025

❑ Why Laplace Transform?

The Laplace transform provides a very convenient and powerful approach to analysis and controls design for linear time invariant systems. Thus, given a single-input/single-output system with input *u(t)* and output *y(t)*, one may completely characterize its response by knowing either the impulse response *h(t)*, its Laplace transform, or the transfer function *H(s)*.



In the time domain, one has the convolution identity

$$y(t) = h * u(t) \equiv \int_0^t h(t-\tau)u(\tau)\,d\tau = \int_0^t h(\tau)u(t-\tau)\,d\tau$$

In the frequency domain one may simply multiply transforms to obtain the output

$$Y(s) = H(s)U(s)$$

❑ Unit Impulse

L^{-1} → inverse Laplace transform

$y(t) = L^{-1}(Y(s)) = L^{-1}(H(s))$, by setting $U(s) = 1$.

❑ Step Response

$y(t) = L^{-1}(Y(s)) = L^{-1}(H(s)/s)$, by setting $U(s) = 1/s$.

Note: the step response is the integral of the unit impulse response.

Not Y(s)

❑ Poles

- The system *poles* are the roots of the denominator of *H(s)*.

❑ Zeros

- The system *zeros* are the roots of the numerator of *H(s)*.

❑ Natural modes

- The system *natural modes* are the time functions corresponding to the poles. There is one natural mode per pole.

❑ Solution of ordinary differential equation with zero initial conditions

- **Poles, zeros, natural modes, response**

Example 1.

A system is characterized by the ODE

$$y''+3y'+2y=u'-u$$

Find the transfer function, the poles, zeros, and natural modes. Find the impulse response, and the step response. Find the output *y(t)* if all initial conditions are zero and the $u(t)=e^{-3t}u_{-1}(t)$.

$$u(s)= \mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

1. Transfer function

$$Y(s) = H(s)\,U(s) \Rightarrow H(s) = \frac{Y(s)}{U(s)}$$

$$s^2 Y(s) + 3s\,Y(s) + 2Y(s)$$

$$= s\,U(s) - U(s)$$

$$(s^2 + 3s + 2)\,Y(s) = (s-1)\,U(s)$$

$$H(s) = \frac{s-1}{s^2+3s+2}$$

2. poles, zeros

$$s^2+3s+2=0$$

$$(s+1)(s+2)=0$$

$$s=-1 \quad s=-2$$

$$s-1=1$$

$$s=1$$

3. natural modes, $\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$, $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$

e^{-t} , e^{-2t}

4. Find impulse response → time function

$U(s) = 1$ / $\frac{1}{s}$

$Y(s) = H(s)$

$y(t) = L^{-1}(H(s))$

$$L^{-1}\left(\frac{s-1}{s^2+3s+2}\right)$$

Using PFE, find k_1 k_2 in

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$k_1 = -2 \quad , \quad k_2 = 3$$

$$Y(s) = \frac{-2}{s+1} + \frac{3}{s+2}$$

$$y(t) = \left(-2e^{-t} + 3e^{-2t}\right) u(t)$$

5. step response

$$U(s) = \frac{1}{s}$$

$$Y(s) = H(s) \cdot \frac{1}{s}$$

$$= \frac{s-1}{s^2+3s+2} \cdot \frac{1}{s}$$

$$y(t) = \underline{\underline{\mathcal{L}}}^{-1}\left(Y(s) \right)$$

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

Using PFE, $k_1 = -\frac{1}{2}$, $k_2 = 2$, $k_3 = -\frac{3}{2}$ ✓

$$y(t) = \left(-\frac{1}{2} + 2e^{-t} - \frac{3}{2}e^{-2t}\right) u_{-1}(t)$$

6. $u(t) = \underline{e^{-3t}}\,\underline{u_{-1}(t)} \Rightarrow U(s) = \frac{1}{s+3}$

$$Y(s) = \frac{s-1}{s^2+3s+5} \cdot \frac{1}{s+3}$$

$$= \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3}$$

$$k_1 = -1, \quad k_2 = 3, \quad k_3 = -2$$

$$y(t) = \left(-e^{-t} + 3e^{-2t} - 2e^{-3t}\right) u_{-1}(t)$$

❑ Solution of ordinary differential equation with nonzero ICs.

If the initial conditions are nonzero, one must use the more general differentiation property from the Table of Properties

$$L\left(x^{(n)}(t)\right) = s^{n}X(s) - s^{n-1}x(0^{-}) - \cdots - x^{(n-1)}(0^{-})$$

$\to 0 \qquad \to 0$

$L(x') = sX(s) - \boxed{}$

Example 2.

Consider a system described as

$$y''+3y'+2y = u'-u$$

Find the output *y(t)* if the input is $u(t) = e^{-3t}u_{-1}(t)$ and the initial conditions are $y(0) = 1, y'(0) = -1$.

$L(y'') = s^2 Y(s) - s\,y(0) - y'(0)$

$L(y') = s\,Y(s) - y(0)$

$$s^2 Y(s) - s + 1$$

$$+ 3\left(sY(s) - 1 \right) = sU(s) - U(s)$$

$$+ 2Y(s)$$

$$\Rightarrow \left(s^2 + 3s + 2\right) Y(s) - s - 2$$

$$= (s-1)U(s)$$

$$Y(s) = \frac{s+2}{s^2+3s+2} + \frac{s-1}{s^2+3s+2} U(s)$$

from none-zero intial conditions

$$Y(s) = H(s)\,U(s) + \text{intial terms}$$

$$Y(s) = \frac{s+2}{s^2+3s+2} + \frac{s-1}{s^2+3s+2}\,\frac{1}{s+3}$$

$$= \frac{\cancel{s+2}}{(s+1)\cancel{(s+2)}} + \frac{s-1}{(s+1)(s+2)(s+3)}$$

$$= Y_1(s) + Y_2(s)$$

$$Y_1(s) = \frac{1}{s+1}$$

$$y_1(t) = e^{-t}\, u_{-1}(t)$$

$$Y_2(s) = \frac{s-1}{(s+1)(s+2)(s+3)}$$

$$= \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = (s+1)\, Y_2(s)\Big|_{s=-1} = \frac{s-1}{(s+2)(s+3)}\Big|_{s=-1}$$

$$= \frac{-2}{1 \cdot 2} = -1$$

$$k_2 = (s+2)\, Y_2(s)\Big|_{s=-2} = \frac{s-1}{(s+1)(s+3)}\Big|_{s=-2} = \frac{-3}{-1 \cdot 1}$$

$$= 3$$

$$k_3 = (s+3)\, Y_2(s)\Big|_{s=-3} = \frac{s-1}{(s+1)(s+2)}\Big|_{s=-3} = \frac{-4}{-2 \cdot (-1)}$$

$$= -2$$

$$Y_2(s) = \frac{-1}{s+1} + \frac{3}{s+2} + \frac{-2}{s+3}$$

$$y_2(t) = \left(-e^{-t} + 3e^{-2t} - 2e^{-3t}\right) u_{-1}(t)$$

$$y(t) = y_1(t) + y_2(t)$$

$$= \left(e^{-t} - e^{-t} + 3e^{-2t} - 2e^{-3t}\right) u_{-1}(t)$$

$$= \left(3e^{-2t} - 2e^{-3t}\right) u_{-1}(t) \quad //$$