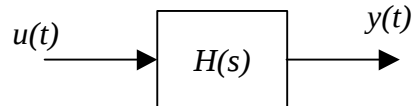

TRANSFER FUNCTION AND ODE SOLUTION

TRANSFER FUNCTION, POLES, ZEROS, STEP RESPONSE

The Laplace transform provides a very convenient and powerful approach to analysis and controls design for linear time invariant (LTI) systems. Thus, given a single-input/single-output (SISO) system with input $u(t)$ and output $y(t)$, one may completely characterize its response by knowing either the impulse response $h(t)$ or its Laplace transform, the transfer function $H(s)$.



In fact, in the time domain one has the convolution identity

$$y(t) = h * u(t) \equiv \int_0^t h(t-t)u(t) dt = \int_0^t h(t)u(t-t) dt .$$

The limits are a consequence of the fact that both the system $h(t)$ and the input $u(t)$ are causal (e.g. equal to zero for $t < 0$).

In the frequency domain one may simply multiply transforms to obtain the output $Y(s) = H(s)U(s)$. (1)

One notes that this equation only holds if the initial conditions (ICs) are equal to zero. Otherwise, there is an additional IC term added.

Recall that the transfer function is the Laplace transform of the impulse response $H(s) = L(h(t))$.

This is easily seen by setting $U(s)=1$, corresponding to the unit impulse, in (1). To find the *step response* $r(t)$ one sets $U(s)=1/s$ in (1) to see that

$$R(s) = \frac{H(s)}{s}.$$

Note that the step response is the integral of the impulse response. (Recall the integration property of Laplace transforms.)

The system *poles* are the roots of the denominator of $H(s)$. The system *zeros* are the roots of the numerator of $H(s)$. The *system natural modes* are the time functions corresponding to the poles. There is one natural mode per pole.

The next example makes the point that an LTI system can also be characterized by an ordinary differential equation (ODE), and reviews the notions of poles, zeros, natural modes, and step response.

Example 1- Solution of ODE with Zero ICs, Poles, Zeros, Natural Modes, Step Response

A system is characterized by the ordinary differential equation (ODE)
 $y''+3y'+2y = u'-u$.

Find the transfer function. Find the poles, zeros, and natural modes. Find the impulse response. Find the step response. Find the output $y(t)$ if all ICs are zero and the input is $u(t) = e^{-3t}u_{-1}(t)$.

a. Transfer Function

First one transforms the ODE to obtain
 $s^2Y(s) + 3sY(s) + 2Y(s) = sU(s) - U(s),$

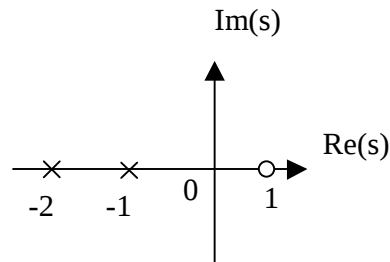
whence one may write the transfer function

$$H(s) \equiv \frac{Y(s)}{U(s)} = \frac{s-1}{s^2+3s+2} = \frac{s-1}{(s+1)(s+2)}.$$

b. Poles, Zeros, Natural Modes

The poles are at $s = -1, -2$ and the zero is at $s=1$. Recall that the number of poles and zeros must be the same, so that there is also one zero at infinity. In fact, note that $H(s)$ goes to zero as s blows up to infinity since the relative degree is one.

One denotes the poles and zeros by drawing the pole/zero plot in the s -domain.



The natural modes are e^{-t}, e^{-2t} .

c. Impulse Response

The impulse response is found by inverse transforming $H(s)$. One performs the PFE to obtain

$$H(s) = \frac{s-1}{(s+1)(s+2)} = \frac{-2}{s+1} + \frac{3}{s+2},$$

whence the transform tables yield

$$h(t) = [-2e^{-t} + 3e^{-2t}]u_{-1}(t).$$

Note that the impulse response is always a weighted sum of the natural modes. The coefficients depend on the pole and zero locations.

d. Step Response

The step response is found by inverse transforming $H(s)/s$. One performs the PFE to obtain

$$\frac{H(s)}{s} = \frac{s-1}{s(s+1)(s+2)} = \frac{-1/2}{s} + \frac{2}{s+1} + \frac{-3/2}{s+2},$$

whence the transform tables yield

$$h(t) = [-\frac{1}{2} + 2e^{-t} - \frac{3}{2}e^{-2t}]u_{-1}(t).$$

Note that the step response is a weighted sum of the natural modes plus a constant term. The coefficients depend on the pole and zero locations.

e. Find Output

One has $U(s) = \frac{1}{s+3}$, so that

$$Y(s) = \frac{s-1}{s^2+3s+2} \cdot \frac{1}{s+3}.$$

Now factor the denominator and perform the PFE so that

$$Y(s) = \frac{s-1}{(s+1)(s+2)(s+3)} = \frac{-1}{s+1} + \frac{3}{s+2} + \frac{-2}{s+3}.$$

The transform tables now yield

$$y(t) = \left[-e^{-t} + 3e^{-2t} - 2e^{-3t} \right] \mu_{-1}(t).$$

Note that the output consists of two portions. One part is due to the natural system response and consists of the natural modes. The other part (which may in certain pathological cases be absent in some examples) consists of the frequencies of the input.

Example 2- Solution of ODE with Nonzero ICs

If the initial conditions are nonzero, one must use the more general differentiation property from the Table of Properties

$$L\left(x^{(n)}(t)\right) = s^n X(s) - s^{n-1}x(0^-) - \dots - x^{(n-1)}(0^-),$$

which includes the ICs.

A system is characterized by the ordinary differential equation (ODE)

$$y'' + 3y' + 2y = u' - u.$$

Find the output $y(t)$ if the input is $u(t) = e^{-3t}u_{-1}(t)$ and the initial conditions are $y(0) = 1, y'(0) = -1$.

To accomplish this, transform the ODE including the IC terms to obtain

$$(s^2 Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = sU(s) - U(s).$$

Note that the ICs of the input are not included. Now one has

$$(s^2 + 3s + 2)Y(s) = (sy(0) + y'(0) + 3y(0)) + (s-1)U(s)$$

$$Y(s) = \frac{(sy(0) + y'(0) + 3y(0))}{s^2 + 3s + 2} + \frac{(s-1)}{s^2 + 3s + 2}U(s) = IC \text{ term} + H(s)U(s).$$

It is important to realize that, if the ICS are not zero, there is an extra term added to the equation $Y(s) = H(s)U(s)$.

Now one has

$$Y(s) = \frac{s+2}{(s+1)(s+2)} + \frac{(s-1)}{(s+1)(s+2)(s+3)} = \frac{1}{(s+1)} + \frac{(s-1)}{(s+1)(s+2)(s+3)}. \quad (2)$$

The second term is the same as Example 1. Therefore, the output is

$$\begin{aligned} y(t) &= [e^{-t}]u_{-1}(t) + [-e^{-t} + 3e^{-2t} - 2e^{-3t}]u_{-1}(t) \\ &= [3e^{-2t} - 2e^{-3t}]u_{-1}(t) \end{aligned} \quad (3)$$

In this example one has two interesting phenomena. First, note that the pole and zero factor $(s+2)$ cancels out of the first term in (2). This means that the specified ICs only excite the natural mode e^{-t} and do not excite the mode e^{-2t} . Second, note that the factor e^{-t} cancels out in (3). This means that, for the specified combination of ICs and input, the mode e^{-t} does not even show up in the output.