ELEC 3500
Control Systems
Lecture 3: Inverse Laplace Transform and Partial Fraction Expansion

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❑ Inverse Laplace Transform

Given a time function *f(t)*, its unilateral Laplace transform is given by

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \, dt$$

where $s = \sigma + j\omega$ is a complex variable. The inverse Laplace transform is a complex integral given by

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s) e^{st} \, ds$$

where the integration is performed along a contour in the complex plane. Since this is tedious to deal with, one usually uses the Cauchy theorem to evaluate the inverse transform using

$$f(t) = \Sigma \, enclosed \; residues \; of \; F(s) e^{st}$$

❑ Inverse Laplace Transform with Partial Fraction Expansion

Problem

$$F(s) \xrightarrow{\text{Laplace Transform Table}} f(t)$$

Given a realistic Laplace transform with several poles and zeros, it is not likely to find its time domain solutions in the table. Therefore, we break a transform down as into sum of simpler transforms that are in the table by using the *Partial Fraction Expansion (PFE)*.

❑ Partial Fraction Expansion (PFE)

- Non-Repeated Poles
- Repeated Real Poles
- Complex Pole Pair
- Relative Degree Equal to Zero

Case 1 Non-Repeated Poles

Give a rational polynomial function of s

$$Y(s) = \frac{n(s)}{(s+p_1)d(s)}$$

Relative degree = degree of denominator − degree of numerator

$\geqslant 1$

$$Y(s) = \frac{K_1}{\underline{s+p_1}} + \underline{Y_1(s)}$$

K_1 is the residue of pole at $s=-p_1$

$Y_1(s)$ doesn't have $(s+p_1)$

Example 1- Non-Repeated Poles

$$Y(s) = \frac{s}{s^3 + 6s^2 + 11s + 6}$$

Factor $s^3 + 6s^2 + 11s + 6$

$s(\underline{s^2 + 6s + 11}) + 6$

$s(\underline{s^2 + 6s + 8} + 3)$

$\underline{s(s+2)(s+4)} + 3s + 6$

$+ \underline{3(s+2)}$

$= (s+2)(s(s+4) + 3)$

$s^2 + 4s + 3$

$= (s+2)((s+1)(s+3))$

$= (s+2)(s+1)(s+3)$

$k_1 = (s+p_1)Y(s)\Big|_{s=-p_1}$

$= k_1 + \underbrace{(s+p_1)Y_1(s)}_{\Downarrow\, 0}\Big|_{s=-p_1}$

Example 1- Non-Repeated Poles

$$Y(s) = \frac{s}{s^3 + 6s^2 + 11s + 6}$$

$$Y(s) = \frac{s}{(s+1)(s+2)(s+3)} \quad \leftarrow (s+2)$$

$$= \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = (s+1)\,Y(s)\Big|_{s=-1} = \frac{s}{(s+2)(s+3)}\Big|_{s=-1} = -\frac{1}{2}$$

Example 1- Non-Repeated Poles

$$Y(s) = \frac{s}{s^3 + 6s^2 + 11s + 6}$$

$$k_2 = (s+2)Y(s)\Big|_{s=-2} = \frac{s}{(s+1)(s+3)}\Big|_{s=-2}$$

$$= \frac{-2}{-1 \cdot 1} = 2$$

$$k_3 = (s+3)Y(s)\Big|_{s=-3} = \frac{\overset{-3}{s}}{\underset{-2}{(s+1)}\underset{-1}{(s+2)}}\Big|_{s=-3} = \frac{-3}{2}$$

Example 1- Non-Repeated Poles

$$Y(s) = \frac{s}{s^3 + 6s^2 + 11s + 6}$$

$$Y(s) = \frac{-\frac{1}{2}}{s+1} + \frac{2}{s+2} + \frac{-\frac{3}{2}}{s+3}$$

$$y(t) = \left(-\frac{1}{2}e^{-t} + 2e^{-2t} - \frac{3}{2}e^{-3t}\right)u_{-1}(t)$$

$$u_{-1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Case 2 Repeated Real Poles

Give a rational polynomial function of s with relative degree of at least 1

$$Y(s) = \frac{n(s)}{(s+p_1)^n d(s)}$$

Spile it into

$$Y(s) = \frac{k_{11}}{(s+p_1)^n} + \frac{k_{12}}{(s+p_1)^{n-1}} + \cdots + \frac{k_{1n}}{s+p_1}$$

$$+ Y_1(s)$$

$Y_1(s)$ doesn't have $(s+p_1)$

Example 2- Repeated Real Poles

$$Y(s) = \frac{s}{s^3 + 4s^2 + 5s + 2}$$

$$k_{11} = (s+p_1)^n Y(s) \Big|_{s=-p_1}$$

$k_{12} \cdots k_{1n}$ can't do this

$$Y(s) = \frac{s}{(s+1)^2 (s+2)}$$

$$Y(s) = \frac{k_{11}}{(s+1)^2} + \frac{k_{12}}{s+1} + \frac{k_2}{s+2}$$

$$k_{11} = (s+1)^2 Y(s) \Big|_{s=-1} = \frac{s}{s+2} \Big|_{s=-1} = -1$$

Example 2- Repeated Real Poles

$$Y(s) = \frac{s}{s^3 + 4s^2 + 5s + 2}$$

$$k_2 = (s+2)Y(s)\Big|_{s=-2} = \frac{s}{(s+1)^2}\Big|_{s=-2} = -2$$

$$Y(s) = \frac{-1}{(s+1)^2} + \frac{k_{12}}{s+1} + \frac{-2}{s+2}$$

To evaluate k_{12}, we note $Y(s)$ must hold for all values of s.

Example 2- Repeated Real Poles

$$Y(s) = \frac{s}{s^3 + 4s^2 + 5s + 2}$$

Evaluate at $s=0$

$$Y(0) = \frac{0}{0+2} = \frac{-1}{1} + \frac{k_{12}}{1} + \frac{-2}{2}$$

$$\Downarrow$$

$$k_{12} = 2$$

$$Y(s) = \frac{-1}{(s+1)^2} + \frac{2}{s+1} + \frac{-2}{s+2}$$

Example 2- Repeated Real Poles

$$Y(s) = \frac{s}{s^3 + 4s^2 + 5s + 2}$$

$$y(t) = \left(-1 \cdot t e^{-t} + 2e^{-t} - 2e^{-2t}\right) u_{-1}(t)$$

Case 3 Complex Pole Pair

Give a rational polynomial function of *s* with relative degree of at least 1

$$Y(s) = \frac{n(s)}{((s+\alpha)^2 + \beta^2)d(s)}$$

Example 3- Imaginary Pole Pair

$$Y(s) = \frac{s+1}{s^3 + 2s^2 + 4s + 8}$$

Example 3- Complex Pole Pair

$$Y(s) = \frac{s-2}{(s+1)(s^2+4s+7)}$$

Case 4 Relative Degree Equal to Zero

Let $Y(s)$ have relative degree of zero, then

$$Y(s) = const + Y_1(s)$$

where *const* is a constant and the relative degree of $Y_1(s)$ is one or more.

Example 5- Relative Degree Zero

$$Y(s)=\frac{3s^3+18s^2+34s+18}{s^3+6s^2+11s+6}$$