REVIEW OF LAPLACE TRANSFORM

LAPLACE TRANSFORM

The Laplace transform is very useful in analysis and design for systems that are linear and time-invariant (LTI). Beginning in about 1910, transform techniques were applied to signal processing at Bell Labs for signal filtering and telephone long-lines communication by H. Bode and others. Transform theory subsequently provided the backbone of Classical Control Theory as practiced during the World Wars and up to about 1960, when State Variable techniques began to be used for controls design. Pierre Simon Laplace was a French mathematician who lived 1749-1827, during the age of enlightenment characterized by the French Revolution, Rousseau, Voltaire, and Napoleon Bonaparte.

Given a time function f(t), its unilateral Laplace transform is given by

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\, dt\,,$$

where $s = \sigma + j\omega$ is a complex variable. Different authors may take the lower limit as 0^+ (i.e. not including effects occurring exactly at time t=0) instead of 0^-. The bilateral Laplace transform has a lower limit of $-\infty$ and is not as useful for feedback control theory as the ULT.

One may write

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}\, dt$$

and, comparing this to the Fourier transform, one sees that f(t) may have a Laplace transform though its Fourier transform does not exist. This is due to the fact that the weighted function $f(t)e^{-\sigma t}$ decays faster than $f(t)$ so that its Fourier transform may exist.

The inverse Laplace transform is a complex integral given by

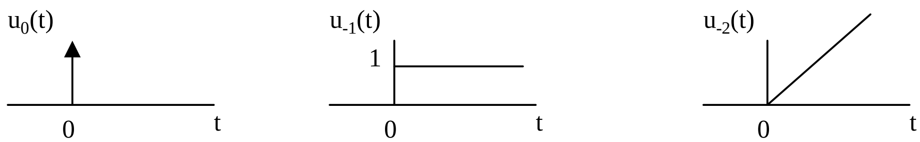
$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}\, ds\,,$$

where the integration is performed along a contour in the complex plane. Since this is tedious to deal with, one usually uses the Cauchy theorem to evaluate the inverse transform using

$$f(t) = \Sigma\, enclosed\ residues\ of\ F(s)e^{st}\ .$$

In this course we shall use lookup tables to evaluate the inverse Laplace transform.

In this course we shall use following notation for the unit test signals:



Unit impulse   Unit step   Unit ramp

Note that each function is the integral of the previous function.

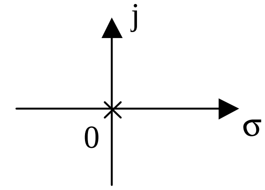
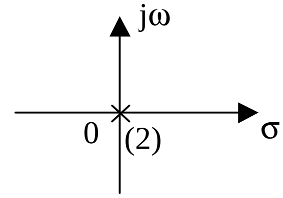
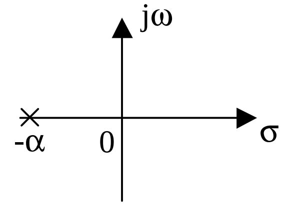
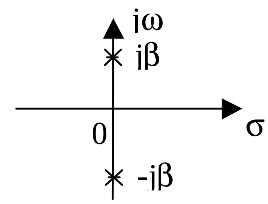
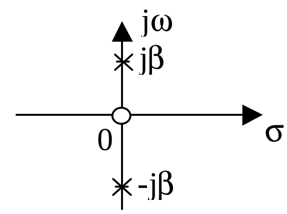
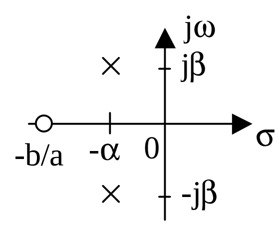
An abbreviated table of Laplace transforms is given here. The text has a more detailed table. Note that we are dealing with the 1-sided transform so that all time functions should be considered to be multiplied by the unit step.

It is important to represent a complex pair of poles in a good way. We shall not split a complex pole pair into two single roots using 'j'. Instead, we shall write $(s+a)^2 + b^2$. Note that this has roots where:

$$\begin{aligned} (s+a)^2 + b^2 &= 0 \\ (s+a)^2 &= -b^2 \\ s+a &= \pm jb \\ s &= -a \pm jb \end{aligned}$$

We shall discuss the complex pole pair in depth when we discuss system performance and second-order systems.

Table of Laplace Transforms

Time Function	Transform	Pole/Zero Plot
$u_0(t)$	1	
$u_{-1}(t)$	$\frac{1}{s}$	
$u_{-2}(t)$	$\frac{1}{s^2}$	
$t^n u_{-1}(t)$	$\frac{n!}{s^{n+1}}$	
e^{-at}	$\frac{1}{s+a}$	
$\sin bt$	$\frac{b}{s^2+b^2}$	
$\cos bt$	$\frac{s}{s^2+b^2}$	
$e^{-at}\left(a\cos bt + \left(\frac{b-aa}{b}\right)\sin bt\right)$	$\frac{as+b}{(s+a)^2+b^2}$	

PROPERTIES OF THE LAPLACE TRANSFORM

Several properties of the Laplace transform are important for system theory. Thus, suppose the transforms of $x(t), y(t)$ are respectively $X(s), Y(s)$. Then one has the following properties.

Laplace Transform Properties

Time-Domain Operation	Frequency Domain Operation
Linearity property $ax(t)+by(t)$	$aX(s)+bY(s)$
Time scaling property $x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$
Time shifting property $x(t-t_0)$	$X(s)e^{-st_0}$
Frequency shifting property $x(t)e^{-at}$	$X(s+a)$
Convolution property $x(t) * y(t)$	$X(s)Y(s)$
Differentiation Property $x'(t)$	$sX(s)-x(0^-)$
Differentiation Property $x^{(n)}(t)$	$s^n X(s)-s^{n-1}x(0^-)-\cdots-x^{(n-1)}(0^-)$
Integration property $\int_{-\infty}^{t} x(t)\,dt$	$\frac{X(s)}{s}+\frac{1}{s}\int_{-\infty}^{0} x(t)\,dt$

In this table, a superscript enclosed in parenthesis denotes differentiation of that order. An asterisk denotes convolution,

$$x(t) * y(t) = \int_0^t x(t-t)y(t)\,dt = \int_0^t x(t)y(t-t)\,dt\ .$$

The upper and lower convolution limits are as shown since all of our time functions are causal (i.e. multiplied by the unit step) in this course.

Note that, in the integration property there is a correction term in the transform if the time function x(t) is not zero to the left of time t=0. Since all our functions are causal, this term is usually equal to zero in this course.

The time scaling property states the fact that if time is scaled one way, then frequency scales in an opposite manner. For example, if $\lvert a\rvert > 1$ so that the time axis is compressed, then the frequency scale is expanded.

Erwin Schrodinger developed an elegant theory of quantum mechanics using the premise that position and momentum are Laplace transform pairs. This allowed him to show the importance of his famous *wave equation*. In this context, the time scaling property is known as the *Heisenberg Uncertainty Principle*. In fact, it states that if position is known more accurately (i.e. its probability density function is on a compressed scale), then momentum is known less accurately (i.e. its PDF is on an expanded scale). As an interesting final comment, it turns out that in quantum mechanics there are other transform pairs besides position/momentum. One of these is time/energy.

INITIAL AND FINAL VALUE THEOREMS

Two theorems are indispensable in feedback control design.

Initial Value Theorem (IVT).

$$\lim_{s \to \infty} sX(s) = x(0^-) .$$

The IVT says that the (fast) transient behavior of x(t) is determined by the high-frequency content of its spectrum X(s).

Final Value Theorem (FVT).

$$\lim_{s \to 0} sX(s) = \lim_{s \to \infty} x(t) .$$

The FVT says that the steady-state, or DC behavior, of x(t) is determined by the low-frequency content of its spectrum X(s).

Example

Let a time function x(t) have the transform $X(s) = \dfrac{-s^2+3}{s^3+4s^2+3s}$. Using the IVT and the FVT one can determine the initial and final values of x(t) *without ever having to find the function x(t) itself*. This is very important for quick design insight later in the course.

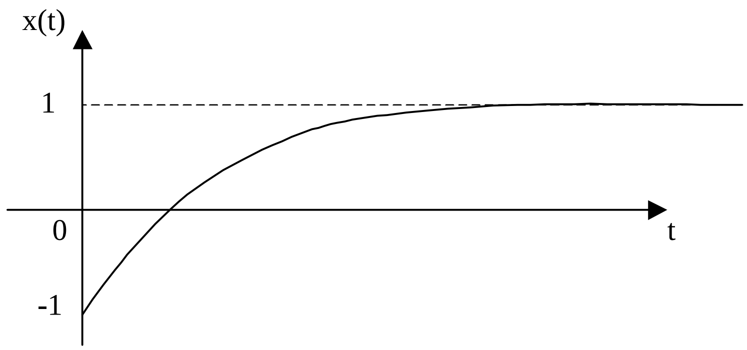
Both the IVT and the FVT rely on the quantity

$$sX(s) = \frac{-s^3+3s}{s^3+4s^2+3s} .$$

According to the IVT, taking the limit of sX(s) as s goes to infinity yields the initial value of $x(0^-) = -1$. To take this limit note that the highest-order terms in the numerator and denominator dominate.

According to the FVT, taking the limit of sX(s) as s goes to zero yields the final value for x(t) of 1. To take this limit note that the lowest-order terms in the numerator and denominator dominate.

Taking the inverse transform (which we shall cover soon) of X(s) yields $x(t) = (1 - e^{-t} - e^{-3t})u_{-1}(t)$. A sketch of this function is given below.



INVERSE LAPLACE TRANSFORM

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where $s = \sigma + j\omega$ is a complex variable. The inverse Laplace transform is a complex integral given by

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}\, ds \,,$$

where the integration is performed along a contour in the complex plane. Since this is tedious to deal with, one usually uses the Cauchy theorem to evaluate the inverse transform using

$$f(t) = \Sigma \, enclosed\ residues\ of\ F(s)e^{st} \,.$$

In this course we shall use lookup tables to evaluate the inverse Laplace transform. An abbreviated table of Laplace transforms was given in the previous lecture. The text has a more detailed table. Given a realistic Laplace transform with several poles and zeros, it is not likely to be contained in the table. However, it is easy to break a transform down as into sum of simpler transforms that are in the table by using the *Partial Fraction Expansion (PFE).*

PARTIAL FRACTION EXPANSION (PFE)

The PFE is simply a technique for splitting a rational transform up into a sum of simpler terms.

Case 1. Non-Repeated Poles

Let a rational polynomial function of s be given as

$$Y(s) = \frac{n(s)}{(s+p_1)d(s)}$$

where *n(s)* is the numerator and *d(s)* is a polynomial which does not have any poles at $s=-p_1$. The *relative degree*, defined as the degree of the denominator (e.g. $(s+p_1)d(s)$) minus the degree of the numerator *n(s)*, should be at least 1.

One may split out a term with divisor $(s+p_1)$ and write

$$Y(s) = \frac{n(s)}{(s+p_1)d(s)} = \frac{K_1}{(s+p_1)} + Y_1(s)$$

where K_1 is known as the *residue of the pole at* $s=-p_1$, and $Y_1(s)$ is a polynomial fraction that does not have any poles at $s=-p_1$. To determine the residue K_1 one may write

$$(s+p_1)Y(s) = \frac{n(s)}{d(s)} = K_1 + (s+p_1)Y_1(s) .$$

Now evaluating all terms at $s=-p_1$, the last term becomes zero and one obtains

$$K_1 = (s+p_1)Y(s)\Big|_{s=-p_1}$$

Example 1- Non-Repeated Poles

Find the inverse Laplace transform of

$$Y(s) = \frac{s}{s^3+6s^2+11s+6} .$$

First note that this function is not in the transform table. To write it as the sum of terms that are in the table, factor the denominator to obtain

$$Y(s) = \frac{s}{(s+1)(s+2)(s+3)} .$$

Now write

$$Y(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

and evaluate the residues by

$$K_1 = \left.\frac{s}{(s+2)(s+3)}\right|_{s=-1} = \frac{(-1)}{(-1+2)(-1+3)} = -\frac{1}{2}$$

$$K_2 = \left.\frac{s}{(s+1)(s+3)}\right|_{s=-2} = \frac{(-2)}{(-2+1)(-2+3)} = 2$$

$$K_3 = \left.\frac{s}{(s+1)(s+2)}\right|_{s=-3} = \frac{(-3)}{(-3+1)(-3+2)} = -\frac{3}{2}$$

Therefore

$$Y(s) = \frac{-1/2}{s+1} + \frac{2}{s+2} + \frac{-3/2}{s+3}.$$

Check this now by combining the terms back into one fraction to get the original *Y(s).*

Each term is now in the table, whence one may write down the inverse Laplace transform as

$$y(t) = (-\frac{1}{2}e^{-t} + 2e^{-2t} - \frac{3}{2}e^{-3t})u_{-1}(t).$$

We multiplied by the unit step to make this function causal.

Case 2- Repeated Real Poles

Let a rational polynomial function of *s* with relative degree of at least 1 be given as

$$Y(s) = \frac{n(s)}{(s+p_1)^n d(s)}$$

where n(s) is the numerator and d(s) is a polynomial which does not have any poles at $s = -p_1$. This has a pole repeated *n* times at $s = -p_1$.

One may show that

$$Y(s) = \frac{n(s)}{(s+p_1)^n d(s)} = \frac{K_{11}}{(s+p_1)^n} + \frac{K_{12}}{(s+p_1)^{n-1}} + \cdots + \frac{K_{1n}}{(s+p_1)} + Y_1(s)$$

where K_{11} is the residue of the pole at $s = -p_1$, and $Y_1(s)$ is a polynomial fraction that does not have any poles at $s = -p_1$. Note that, for a pole of order *n*, one requires in the PFE all terms of order *n* and below.

To determine the residue K_{11} one may proceed as above and use

$$K_{11} = (s+p_1)^n Y(s)\Big|_{s=-p_1}.$$

The other numerator terms K_{ij} can be determined using some derivative formulas given in the text. An alternative technique is given in the next example.

Example 2- Repeated Real Pole

Find the inverse Laplace transform of

$$Y(s) = \frac{s}{s^3 + 4s^2 + 5s + 2}.$$

First, factor the denominator to obtain

$$Y(s) = \frac{s}{(s+1)^2(s+2)}.$$

This may be written as the sum of terms

$$Y(s) = \frac{s}{(s+1)^2(s+2)} = \frac{K_{11}}{(s+1)^2} + \frac{K_{12}}{s+1} + \frac{K_2}{s+2}. \qquad (1)$$

The residues K_{11} and K_2 are determined using the technique in Example 1 to obtain

$$Y(s) = \frac{-1}{(s+1)^2} + \frac{K_{12}}{s+1} + \frac{-2}{s+2}.$$

To evaluate the term K_{12}, one notes that the equality (1) must hold *for all values of s*. Select therefore a convenient value of s, evaluate (1), and solve for K_{12}. A very easy value is $s=0$, for which

$$Y(0) = \frac{0}{(0+1)^2(0+2)} = \frac{-1}{(0+1)^2} + \frac{K_{12}}{0+1} + \frac{-2}{0+2},$$

yielding $K_{12} = 2$.

Therefore, the PFE is

$$Y(s) = \frac{-1}{(s+1)^2} + \frac{2}{s+1} + \frac{-2}{s+2},$$

whence the Laplace transform table gives the time function

$$y(t) = (2e^{-t} - te^{-t} - 2e^{-2t})u_{-1}(t).$$

Note that one may not select a value s corresponding to a pole in using (1) to determine K_{12}, since at the pole values $Y(s)$ blows up to infinity.

Case 3- Complex Pole Pair

Let a rational polynomial function of *s* with relative degree of at least 1 be given as

$$Y(s) = \frac{n(s)}{((s+a)^2 + b^2)d(s)}$$

where n(s) is the numerator and d(s) is a polynomial which does not have the factor $((s+a)^2 + b^2)$.

We have seen in the previous lecture that

$$((s+a)^2 + b^2) = \left(s+(a+jb)\right)\left(s+(a-jb)\right),$$

so this factor corresponds to a complex pole pair at $s = -a \pm jb$. In this course we shall not write factors using 'j'. We shall keep complex pole pairs together in the quadratic form.

One may split out a term with divisor $((s+a)^2 + b^2)$ and write

$$Y(s) = \frac{n(s)}{((s+a)^2 + b^2))d(s)} = \frac{as+b}{(s+a)^2 + b^2)} + Y_1(s)$$

where $Y_1(s)$ is a polynomial fraction that does not contain the factor $((s+a)^2 + b^2)$. Note that the term above the quadratic factor has degree one less, namely, in this case, one.

The next example shows how to solve for the constants *a* and *b*.

Example 3- Imaginary Pole Pair

Find the inverse Laplace transform of

$$Y(s) = \frac{s+1}{s^3 + 2s^2 + 4s + 8}.$$

Factor the denominator and write the PFE to obtain

$$Y(s) = \frac{s+1}{(s+2)(s^2+4)} = \frac{K_1}{s+2} + \frac{as+b}{s^2+4} = \frac{-1/8}{s+2} + \frac{as+b}{s^2+4}, \qquad (2)$$

where the residue at *s=-2* was determined as in Example 1.

To find *a* and *b*, select *two* values of *s* and evaluate both sides of (2). From the resulting algebraic equations, one may solve for *a* and *b*. If *s=0* is not a pole, one may select *s=0* and find *b* first, then find *a* using another value of *s*. If *s=0* is a pole, one must select two nonzero values of *s*, which generally give two simultaneous equations for *a* and *b*.

Since zero is not a pole in this example, evaluate (2) at *s=0* to obtain

$$Y(0) = \frac{0+1}{(0+2)(0^2+4)} = \frac{-1/8}{0+2} + \frac{a.0+b}{0^2+4}.$$

Solving yields *b=3/4*.

Selecting now *s=1* gives

$$Y(1) = \frac{1+1}{(1+2)(1^2+4)} = \frac{-1/8}{1+2} + \frac{a+3/4}{1^2+4},$$

which yields *a=1/8*.

The PFE is thus given by

$$Y(s) = \frac{s+1}{(s+2)(s^2+4)} = \frac{-1/8}{s+2} + \frac{1/8\, s + 3/4}{s^2+4} = \frac{-1/8}{s+2} + \frac{1/8\, s}{s^2+4} + \frac{3/4}{s^2+4}.$$

Now the transform table says that

$$y(t) = \left(-\frac{1}{8}e^{-2t} + \frac{1}{8}\cos 2t + \frac{3}{8}\sin 2t \right) u_{-1}(t).$$

Note that the poles are at $s = -2, \pm j2$. However, we did not use any terms in 'j' in this evaluation. We kept the imaginary pole pair together as $s^2 + b^2$.

Example 4- Complex Pole Pair

Find the inverse transform of

$$Y(s) = \frac{s-2}{(s+1)(s^2+4s+7)}.$$

Using the techniques in Example 3 one may see that

$$Y(s) = \frac{-3/4}{s+1} + \frac{3/4\, s + 13/4}{s^2+4s+7}.$$

To find the inverse transform, write the denominator in the form $(s+a)^2 + b^2$ to get

$$Y(s)=\frac{-3/4}{s+1}+\frac{3/4\,s+13/4}{(s+2)^2+3}.$$

Now use the table to see that

$$y(t)=\left[\frac{-3}{4}e^{-t}+e^{-2t}\left(\frac{3}{4}\cos\sqrt{3}t+\frac{7}{4\sqrt{3}}\sin\sqrt{3}t\right)\right]u_{-1}(t).$$

Note that the complex poles are at $s=-2\pm j\sqrt{3}$, yet we never use 'j' in this evaluation. We are careful to keep the complex pair together in the quadratic term $(s+a)^2+b^2$. Note further that the exponential decay term of the complex pair is e^{-at}, while the frequency of oscillation is given by $b=\sqrt{3}$.

Case 4-Relative Degree Equal to Zero

In control theory we do not use Laplace transforms with relative degree less than zero. If the relative degree of *Y(s)* is equal to zero, then prior to using the above PFE techniques one must do one step of long division to write

$$Y(s)=const+Y_1(s)$$

where *const* is a constant and the relative degree of $Y_1(s)$ is one or more.

Example 5- Relative Degree Zero

Find the inverse Laplace transform of

$$Y(s)=\frac{3s^3+18s^2+34s+18}{s^3+6s^2+11s+6}.$$

This transform has relative degree of zero, so the PFE does not give the correct answer. To find the time function, perform one step of long division to write

$$Y(s)=3+\frac{s}{s^3+6s^2+11s+6}.$$

The second term now has relative degree of 1 and one can proceed using the techniques given above.

In fact, using the results of Example 1, one sees that

$$y(t)=u_0(t)+(-\frac{1}{2}e^{-t}+2e^{-2t}-\frac{3}{2}e^{-3t})u_{-1}(t).$$

If the transform relative degree is zero, it means that the time function contains the impulse function $u_0(t)$.