

ELEC 3500
Control Systems
Lecture 2: State-Variable Systems and
Simulation using Matlab

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□ **State-Variable Systems**

- State-variable systems
- Differential equations to state-variable systems
- Examples

□ **Matlab for Solving State-Variable Systems**

- Matrix operations and indexing
- Ode functions with examples

- State-variable systems $\Rightarrow$ state-space representation

A state-variable system models a dynamic system and comprises a first-order ordinary differential system (ODE) that describes time derivatives of a set of state variables.

The most general state-variable system of a linear system with p inputs, q outputs and n state variables is written in the following form

$$\begin{aligned} \frac{dx}{dt} & \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x' & \quad y(t) = C(t)x(t) + D(t)u(t) \end{aligned}$$

$x(t)$ state variable vector $\in \mathbb{R}^n$ $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
 $y(t)$ output variable vector $\in \mathbb{R}^q$ $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}$
 $u(t)$ control input variable vector $\in \mathbb{R}^p$ $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$

$\dot{x}$ $\frac{dx}{dt}$
 x'

$A(t)$ state matrix $\in \mathbb{R}^{n \times n}$

$B(t)$ input matrix $\in \mathbb{R}^{n \times p}$

$C(t)$ output matrix $\in \mathbb{R}^{q \times n}$

$D(t)$ feedforward matrix $\in \mathbb{R}^{q \times p}$

(A, B, C, D) : time invariant system

$(A(t), B(t), C(t), D(t))$: time variant system

Discrete-time systems

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

$$y(k+1) = C(k)x(k) + D(k)u(k)$$



linear systems

$$\dot{x} = x$$

$$\dot{x} = x^2$$

$$\dot{x}(t) = f(x(t), u(t))$$



nonlinear systems

$$y(t) = h(x(t), u(t))$$

- Ordinary differential equations to state-variable systems

Example: Newton's second law

$$m \ddot{x} = F(x(t))$$

$$m \ddot{x} = F(x(t))$$

Define $z_1 = x$ $\Rightarrow$ $z_1' = z_2$

$z_2 = \dot{x}$ $\Rightarrow$ $z_2' = \frac{F}{m}$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{F}{m}$$

- Ordinary differential equations to state-variable systems

In general

$$y^{(n)} + \alpha_{n-1} y^{(n-1)} + \dots + \alpha_1 y' + \alpha_0 y = 0$$

coefficients

Step 1: define new variables starting with y
No. of $\downarrow$ is the order of ODE

Step 2: Express the derivative of new variables
in term of new variables

- Ordinary differential equations to state-variable systems

step 1:

$$\begin{aligned} z_1 &= y \\ z_2 &= y' \\ &\vdots \\ z_n &= y^{(n-1)} \end{aligned}$$

$$\begin{aligned} y^{(n)} &= -\underline{a_{n-1}} y^{(n-1)} - \underline{a_{n-2}} y^{(n-2)} \\ &\quad \dots - \underline{a_1} y' - \underline{a_0} y \end{aligned}$$

step 2:

$$\begin{aligned} \dot{z}_1 &= \dot{y} = z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= \underline{y^{(n)}} = -a_{n-1} z_n - a_{n-2} z_{n-1} \dots - a_1 z_2 - a_0 z_1 \end{aligned}$$

- Examples

If $n = 3$

$$\left\{ \begin{array}{l} \dot{z}_1 = z_2 \quad \checkmark \\ \dot{z}_2 = z_3 \quad \checkmark \\ \dot{z}_3 = -\alpha_2 z_3 - \alpha_1 z_2 - \alpha_0 z_1 \end{array} \right\}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \mathbf{A} \mathbf{z} \quad \checkmark$$

- Examples

State-variable system for general ODE is

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = z_3$$

$$\vdots$$

$$\dot{z}_n = -\alpha_{n-1} z_n - \alpha_{n-2} z_{n-1} \dots - \alpha_1 z_2 - \alpha_0 z_1$$

- Examples

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$\downarrow \quad \quad \quad \Downarrow \quad \quad \quad \leftarrow$
 $\dot{z} = A z$

- Examples

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 0$$

Step 1

$$\begin{cases} z_1 = y \\ z_2 = \dot{y} \end{cases}$$

$$\ddot{y} = -\dot{y} - y$$

Step 2

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \ddot{y} = -\underline{z_2} - \underline{z_1}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

- Examples

$$\dot{y}(t) + \underbrace{y(t)^2 \dot{y}(t)} + y(t) = 0$$

$$z_1 = y$$

$$z_2 = \dot{y}$$

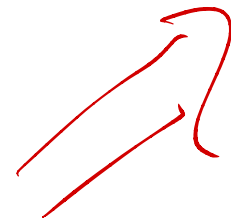
$$\dot{z}_1 = \dot{y} = z_2$$

$$\dot{z}_2 = -y^2 \dot{y} - y = -z_1^2 z_2 - z_1$$

no y

$$\dot{z}_1 = z_2$$

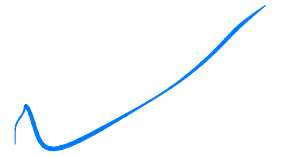
$$\dot{z}_2 = -z_1^2 z_2 - z_1$$



- Examples

2-DOF Robotic Arm Model

$$M\ddot{q}(t) + \underbrace{ML\dot{q}^2(t) + MgL\sin q(t)} = u(t)$$



M : mass

$q(t)$: time-vary angle

L : Lagrange constant

g : gravitational acceleration

$u(t)$: force

$$\ddot{q} = -L\dot{q}^2(t)$$

$$-gL\sin q(t) + \frac{u(t)}{M}$$

- Examples

Define $x_1 = q$

$x_2 = \dot{q}$

$\dot{x}_1 = x_2$ ✓

$$\begin{cases} \dot{x}_2 = \ddot{q} = -L x_2^2 - g L \sin x_1 + \frac{u(t)}{m} \end{cases}$$

$\Downarrow$

✓ $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -L x_2^2 - g L \sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$