ELEC 3500
Control Systems
Lecture 2: State-Variable Systems and Simulation using Matlab

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❑ State-Variable Systems

- State-variable systems
- Differential equations to state-variable systems
- Examples

❑ Matlab for Solving State-Variable Systems

- Matrix operations and indexing
- Ode functions with examples

- State-variable systems $\Rightarrow$ state-space representation

A state-variable system models a dynamic system and comprises a first-order ordinary differential system (ODE) that describes time derivatives of a set of state variables.

The most general state-variable system of a linear system with *p* inputs, *q* outputs and *n* state variables is written in the following form

$\frac{dx}{dt}$ x'

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

$\dot{x}$ $\frac{dx}{dt}$ x'

$x(t)$ state variable vector $\in R^n$ $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$y(t)$ output variable vector $\in R^q$ $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}$

$u(t)$ control input variable vector $\in R^p$ $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$

$A(t)$ state matrix $\in R^{n \times n}$

$B(t)$ input matrix $\in R^{n \times p}$

$C(t)$ output matrix $\in R^{q \times n}$

$D(t)$ feedforward matrix $\in R^{q \times p}$

(A, B, C, D) : time invariant syseem

$(A(t), B(t), C(t), D(t))$: time variant syseem

Discrete-time systems

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k)$$

$$y(k+1) = C(k)\,x(k) + D(k)\,u(k)$$

$$\dot{x} = x$$

$\Uparrow$ linear systems

$$\dot{x} = x^2$$

$\Downarrow$ nonlinear systems $\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$

- Ordinary differential equations to state-variable systems

Example: Newton's second law

$$m\, x^{(2)} = F(x(t))$$

$$m\, x'' = F(x(t))$$

Define $z_1 = x$ $\Rightarrow$ $z_1' = z_2$

$z_2 = x'$ $\qquad$ $z_2' = \frac{F}{m}$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{F}{m}$$

- Ordinary differential equations to state-variable systems

In general

$$\overbrace{y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 y' + \alpha_0 y}= 0$$

coefficients

Step 1: define new variables starting with y

No. of is the order of ODE

Step 2: Express the derivative of new variables in term of new variables

- Ordinary differential equations to state-variable systems

step 1:

$z_1 = y$

$z_2 = y'$

$\vdots$

$z_n = y^{(n-1)}$

$y^{(n)}$

$= -\alpha_{n-1} y^{(n-1)} - \alpha_{n-2} y^{(n-2)}$

$\cdots - \alpha_1 y' - \alpha_0 y$

step 2:

$\dot{z}_1 = \dot{y} = z_2$

$\dot{z}_2 = z_3$

$\vdots$

$\dot{z}_n = y^{(n)} = -\alpha_{n-1} z_n - \alpha_{n-2} z_{n-1} \cdots - \alpha_1 z_2 - \alpha_0 z_1$

- Examples

If $n = 3$

$$\begin{cases} \dot{z}_1 = z_2 \checkmark \\ \dot{z}_2 = z_3 \checkmark \\ \dot{z}_3 = -\alpha_2 z_3 - \alpha_1 z_2 - \alpha_0 z_1 \end{cases}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\underline{\dot{z} = A z} \checkmark$$

- Examples

State-variable system for general ODE is

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = z_3$$

$$\vdots$$

$$\dot{z}_n = -\alpha_{n-1} z_n - \alpha_{n-2} z_{n-1} \cdots - \alpha_1 z_2 - \alpha_0 z_1$$

- Examples

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ & & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\dot{z} = A z$$

- Examples

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 0$$

$$\ddot{y} = -\dot{y} - y$$

step 1
$$\begin{cases} z_1 = y \\ z_2 = y' \end{cases}$$

step 2
$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \ddot{y} = -z_2 - z_1$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

- Examples

$$\ddot{y}(t) + \underline{y^2(t)\dot{y}(t) + y(t)} = 0$$

$$z_1 = y$$

$$z_2 = \dot{y}$$

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -z_1^2 z_2 - z_1$$

$$\dot{z}_1 = \dot{y} = z_2$$

$$\dot{z}_2 = -y^2\dot{y} - y = -z_1^2 z_2 - z_1$$

no y

- Examples

2-DOF Robot Arm Model

$$M\ddot{q}(t) + ML\dot{q}^2(t) + MgL\sin q(t) = u(t)$$

M : mass

$q(t)$: time-vary angle

L : Lagrange constant

g : gravitational acceleration

$u(t)$: force

$$\ddot{q} = -L\dot{q}^2(t) - gL\sin q(t) + \frac{u(t)}{M}$$

- Examples

Define $x_1 = q$

$x_2 = \dot{q}$

$$\begin{cases} \dot{x}_1 = x_2 \quad \checkmark \\ \dot{x}_2 = \ddot{q} = -L x_2^2 - gL\sin x_1 + \frac{u(t)}{m} \end{cases}$$

$\Downarrow$

$$\checkmark \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -L x_2^2 - gL\sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) =$$